Simplified Mechanisms with Applications to Sponsored Search and Package Auctions

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First draft: August 16, 2007
This draft: September 23, 2007

A simplified mechanism is a direct mechanism modified by restricting the set of reports or bids. An example is the auction used to place ads on Internet search pages, in which each advertiser bids a single price to determine the allocation of eight or more ad positions on a page. If a simplified mechanism satisfies the "best-reply-closure" property, then all Nash equilibria of the simplified mechanism are also equilibria of the original direct mechanism. For search advertising and package auctions, simplification eliminates certain inefficient, low-revenue equilibria that are favored in the original direct mechanism when bidding costs are positive.

Keywords: simplified mechanisms, sponsored search, generalized second price auctions, package auctions, combinatorial auctions, core-selecting auctions, mechanism design.

***JEL Categories**:* D44, C78

I. Introduction

Real-world auctions are often much simpler than might be expected according to received economic theory, particularly when there are multiple items or lots for sale. Although simplification itself is hardly a surprise, a useful theory of simplified

[1] Support for this research was provided by National Science Foundation grant SES-0648293 and by Yahoo! Thanks to Richard Steinberg for comments. Any opinions expressed here are those of the author alone.

mechanisms can deepen understanding about it, characterizing what simplification means, accounting for the kinds of simplification that are found in practice, and providing guidance on how to simplify proposed new auction mechanisms.

An example of a simplified auction and how theory can help to parse it is the world's most frequently used auction method, which is initiated whenever a user searches at a site like Google or Yahoo! For each search, the site conducts an automated auction to determine the placement of advertisements into multiple positions—currently eight or more—on the search results page. In preparation for these auctions, each bidder names search terms that will trigger its bid and a price per click for each term or group of terms. The auctioneer converts these bid prices per click into prices per impression by multiplying the bid by the estimated "clickability" of the ad, which is essentially the estimated click rate that the ad would experience if it were shown in the first position on the search results page. The ads are shown on the page in the order determined by these prices per impression, beginning with the highest. An advertiser pays only when its ad is actually clicked and then pays only the smallest bid per click that would win the same ad position. Its expected price per impression is therefore the smallest price per impression that would win its position.

If there were only one position on the search page and all bids were directly expressed as prices per impression, then the auction just described would be a standard second-price auction. In the actual situation with its multiple positions, the mechanism has been dubbed the "generalized second-price auction." It is equivalent to a series of second price auctions with separate bids for each position but in which the bid for any position n is restricted to be equal to the bid for the first position multiplied by a factor proportional to the estimated click rate position *n* and with the additional proviso that any bidder who wins a position is excluded from the auctions for lower positions.

In a pair of recent papers, Edelman, Ostrovsky, and Schwartz (2007) and Varian (2006) have studied the generalized second-price auctions using the assumptions that bidders value all clicks equally (regardless of the position of the ad) and that bidder payoffs are equal to the value of their clicks minus the total prices they pay. A central finding of both papers is that the prices and assignments of positions resulting from a selected full-information Nash equilibrium of the generalized second price auction is the same as for the dominant strategy equilibrium of a multi-item Vickrey auction.

This theory leaves several questions about sponsored search auctions unanswered. One is: why do advertisers pay on a per click basis, rather than on the per impression basis that is commonly used for advertisements for print ads, radio and television? In a static full-information environment, there would be little to distinguish between two approaches to pricing, although per click charges are easier for an Internet bidder to audit because it can meter visits to its own site.

In the environment of the Internet, there is a second important advantage to per click pricing. Search companies have continually expanded their scope in various ways, showing ads on a wider variety of sites and encouraging advertisers to use “extended match” technologies to place ads not only on pages that match the bidder’s search term exactly but also on pages that match approximately. As an illustration, the extended search technology might deem the term “ink cartridge” to be sufficiently related to the term “printer cartridge” and might show an ad for the latter when the search is made for the former. The relation among these search terms is imperfect, however, because “ink cartridge" might be entered by a user searching for a pen ink refill, so the proportion of searchers who are potential customers for a printer ink company would be lower for the related term, which makes each impression less valuable. Even on a per click basis, the values may be different, because clicks from pen ink searchers would less frequently result in actual sales. Still, pricing ads on a per click basis reduces the advertiser's cost per impression for ads on less closely related search results pages, which makes more advertisers willing to use the extended search technology. That, in turn, increases the effective demand for and utilization of ad space and raises the search company's revenues and profits.

Another question concerns not the distinction between price-per-click and price-per-impression bids, but the choice of auction rules. If, as the prior literature assumes, the Vickrey outcome is a desirable one, then why not just use the Vickrey auction mechanism instead of the generalized second-price auction?[2] Not only does the Vickrey mechanism implement the desired outcome using dominant strategies rather than merely full-information Nash equilibrium, but it does so for a realistically wider class of environments in which the value of a click may depend on the position of the ad in addition to the search term.

We divide this very general question into two sharper ones by treating the generalized second-price auction as differing from the Vickrey mechanism in two ways: its bids are one-dimensional while those required by the Vickrey mechanism are multidimensional and, given the values that might be inferred from the bids, the pricing

[2] In early postings describing the auction, Google claimed that this generalized second price auction was the actual Vickrey auction, but that is a mistake. In particular, no bidder has dominant strategy in the generalized second price auction.

of the ad positions is determined by a series of separate second price auctions rather than by the Vickrey formula.[3] These two differences suggest two questions, each holding one aspect of the design fixed. First, if the auctioneer has decided to use a series of second price auctions, why does it then accept only a single price per click and impute values to all positions instead of allowing multidimensional bids that state directly all the relevant values? Also, would the same reasons apply if the Vickrey pricing rule had been used? Second, if the auctioneer has decided to use a single, one-dimensional bid and impute values for the different positions, what advantage might it enjoy by using a second-price rule rather than the Vickrey pricing rule?

To answer the first question, we observe that in any series of second-price auctions, it is the losing bids for the various positions that determine the prices. If individual bids for each position were permitted but not required and if there were any arbitrarily small positive cost incurred by a bidder in submitting individual bids, then there would be no pure full-information equilibrium at which the seller earns positive revenue, because losing bidders for a position would never make positive bids.[4] Even when the cost of submitting bids is zero, the series of second price auctions with individual bids still admits some of these same zero-revenue strategy profiles as Nash equilibria. A similar argument applies when the Vickrey pricing rule is used. In contrast, every equilibrium of the generalized second price auction for two or more items has positive revenues, because a bidder whose positive bid is winning for position *n*

[3] Although this represents just one particular way to extend the generalized second price auction to a multidimensional bidding mechanism, it is an especially significant one, because a series of ascending auctions is a common way to sell similar heterogeneous items in off-line settings.

[4] This paper uses full-information Nash equilibrium to analyze various mechanisms. Based on earlier empirical successes and failures of game-theoretic auction models, what we believe should be taken most seriously from this analysis is the *comparative* predictions about the revenue performance of alternative auctions mechanisms, rather than the point predictions about the performance of any single mechanism.

necessarily also determines a positive price for position n–1. We will argue below that this analysis, which seems tailored to exploit the particular structure of the generalized second price auction, nevertheless applies more broadly and illustrates a general principle of mechanism design: certain kinds of simplifications of the strategy set always *reduce* the set of pure Nash equilibria—often by eliminating inefficient or low-revenue equilibria—*without* introducing additional pure equilibria.

For the second question, although the received literature already includes analyses highlighting important disadvantages of the Vickrey pricing formula in multi-item auctions (Ausubel and Milgrom (2005) and Rothkopf (2007)), the most devastating objections apply only to auctions in which bidders can buy multiple items. The objections have no force for sponsored search auctions, because each bidder in such an auction is restricted to buy at most a single position.

Our answer to the second question focuses on the special environment of sponsored search, for which a distinct analysis is needed. We extend the models used in earlier studies to allow heterogeneity among searchers. We assume that there are two kinds of searchers—some are potential customers who are actually looking for a product to buy and others are merely curious about the products being advertised—with each group having its own click rates for ads occupying different positions on the page. For example, it may be that clicks on ads near the bottom of a search page come more frequently from potential customers because these searchers more often attend to the full list of the ads. In that case, if clicks from potential customers are more valuable than clicks from other searchers, then clicks on ads near the bottom of a page will be more valuable than clicks on ads near the top, because a higher proportion of these clicks will come from potential buyers. In general, we need only assume that the click rates for the groups are different to conclude that clicks from different positions have different values.

Our formal model assumes that each advertiser has some positive value per click from potential customers and a zero value per click from other searchers and that the frequency of clicks from each group falls as one moves down the search page. With these assumptions, the bidders' types are one-dimensional and the value per impression is lower in lower positions, just as in the prior literature. Based on the data at its own site, the auctioneer can observe the empirical click rate for each position but not the purchase behavior of clickers once they leave the search page. The auctioneer cannot determine from its own observations and a bidder's reported value for an ad in one position what the bidders' values are for ads in the other positions. Therefore, it cannot use one-dimensional bids to conduct a Vickrey auction despite the one-dimensional type spaces. In contrast, the analyses of the previously cited papers can be generalized somewhat to establish that, if a certain condition is satisfied, there still exists a full-information equilibrium of the generalized second price auction in which the realized prices are Vickrey prices. This is possible because, unlike the auctioneer, each bidder can observe how its clicks from various ad positions convert into sales and profits.

These lessons illustrated by sponsored search auctions point to a more expansive theory of simplified mechanisms derived from direct mechanisms by restricting the set of allowable reports or bids. The key characteristic of successful simplifications is the best-reply-closure property, defined as follows: for any participant *j*, if the other participants play only their own simplified strategies, then participant *j*'s set of simplified strategies includes a best reply to the profile of others' strategies. We show that the simplification of the strategy space for sponsored search auctions satisfies the best-reply closure property. The main general theorem holds that a pure profile of simplified strategies is a pure Nash equilibrium of a simplified mechanism with the best-reply closure property if and only if it is a pure Nash equilibrium of the original mechanism. This means that a suitable simplification can eliminate pure equilibria (by eliminating one or more of the strategies it uses) while otherwise leaving the set of equilibria unchanged.

Besides Internet search advertising, a second significant application of simplified mechanisms is to the problem of *package auctions* (also known as *combinatorial auctions*). These are mechanisms in which there are multiple (often heterogeneous) items for sale and bidders are potentially interested in buying any packages, that is, subsets of the full set of items. With M items for sale and quasi-linear preferences, a full description of a bidder's preferences specifies values for all $2^M - 1$ non-empty packages. If a direct package auction mechanism were attempted for a sale like FCC spectrum auction #66 in which 1122 licenses were offered for sale, a bidder could feasibly compute and report values for only an extremely minute fraction of the roughly 10^{338} available packages. If we model this fact by assuming that bidders can submit a modest number of packages bid at no cost but eventually incur a small cost for each additional package bid, then (as we will show) there are many inefficient and low-revenue equilibria of the full game. A suitably simplified package auction, however, can eliminate many or all of the "undesirable" equilibria without introducing new Nash equilibria.

Our analysis of package bidding is limited to the class of core-selecting package auction mechanisms (see Day and Milgrom (2007) and Ausubel and Milgrom (2002)) which, in particular, includes the important menu auction of Bernheim and Whinston

(1986). The full-information equilibrium outcomes of these all mechanisms include all the bidder-optimal core allocations. For each mechanism in this class, a corresponding simplified auction restricts bidders to report only values in an abstract set *V*. With a set of items *N* for sale, a typical element $v \in V$ is a function $v : 2^N \to \mathbb{R}_+$ with the property that $v(\varnothing) = 0$. For $k > 0$, let $v - k$ denote the value function which assigns to any non-empty package *S* the value $v(S) - k$. We show that if the actual values lie in the set *V* and if $v \in V \Rightarrow v - k \in V$, then the best-reply-closure property is satisfied. Consequently, the Nash equilibria of the *V*-simplified mechanism includes only Nash equilibria of the original mechanism, including the identified equilibria for which the outcomes are bidder-optimal core allocations.

Based on that analysis, we suggest some sets *V* that may be useful for applications in which potential value complementarities arise only from shared fixed costs. One useful property of our sets *V* is that they grow only linearly in the number of items *N*, while the full set of package bids grows exponentially in *N*. We evaluate the performance of these simplified mechanisms in particular environments, including ones in which the actual values lie outside of *V*. This analysis allows us to revisit the difficult question of whether, when and how prices might be useful in package auction design.

The rest of this paper is organized as follows. Section II states and proves the *simplification theorem*, which shows that for general games, simplifications that restrict the strategy set to one satisfying the best-reply-closure property shrinks the set of pure Nash equilibrium profiles. Section III treats the generalized second price auction of sponsored search.[5] Its first subsections shows that, compared to a series of second price auctions with general value reports, the simplification used for the generalized second price auction satisfies the best-reply-closure property and eliminates certain zero revenue Nash equilibria. Its second subsection introduces the model described above with two types of searchers and demonstrates that the selected equilibrium of the generalized second price auction still establishes Vickrey prices, thus extending the results of prior research. Section IV treats package bidding, proving the theorem stated above which identifies a class of simplifications that satisfies the best-reply-closure property. Section V concludes.

II. The Simplification Theorem

Let (N, X, π) be a normal form game, where $X = (X_1, ..., X_N)$.

Definition. A product set of strategy profiles $\hat{X} = \hat{X}_1 \times ... \times \hat{X}_N$ has the *best-reply closure property* in (N, X, π) if for every player *n* and every profile $x_{-n} \in \hat{X}_{-n}$ there exists $x_n \in \hat{X}_n$ such that for all $x_n' \in X_n$. $\pi_n(x_n, \hat{x}_{-n}) \geq \pi_n(x_n', \hat{x}_{-n})$.

When the best-reply closure property holds, a player *n* looking for a response to any opposing pure profile $x_{-n} \in \hat{X}_{-n}$ loses nothing by restricting attention to strategies in $\hat{X}_n$.

[5] Throughout our analysis of auctions, we set aside the possibility of ties. These can be treated by an extension of the equilibrium concept, as suggested by Simon and Zame (1990), or by other devices, but these details do not affect any substantive conclusions.

Theorem 1 (Simplification Theorem). Suppose $\hat{X}$ has the best-reply closure property in (N, X, π). Then, a pure strategy profile $\hat{x} \in \hat{X}$ is a Nash equilibrium of $(N, \hat{X}, \pi)$ *if and only if* it is also a Nash equilibrium of (N, X, π).

Proof. The *if* direction is obvious. For the only if direction, suppose that $\hat{x}$ is not a Nash equilibrium of (N, X, π). Then there is some player *n* that has a profitable deviation from $\hat{x}$, that is, for some $x_n' \in X_n$, $\pi_n(x_n', \hat{x}_{-n}) > \pi_n(\hat{x}_n, \hat{x}_{-n})$. According to the best-reply closure property, there is some $x_n \in \hat{X}_n$ such that $\pi_n(x_n, \hat{x}_{-n}) \geq \pi_n(x_n', \hat{x}_{-n})$. Hence, $\pi_n(x_n, \hat{x}_{-n}) > \pi_n(\hat{x}_n, \hat{x}_{-n})$: $\hat{x}$ is not a Nash equilibrium of $(N, \hat{X}, \pi)$. ♦

The interesting part of the theorem is the *only if* assertion. It says that eliminating strategies while preserving the best-reply closure property does not add new equilibrium strategy profiles and hence does not extend the set of equilibrium outcomes. For applications, the trick is to specify $\hat{X}$ to eliminate the “bad” equilibria while preserving the “good” equilibria and to verify the property, so that no new bad equilibria are introduced.

The simplification theorem has been stated above for equilibria in pure strategies and we will apply it in that form. Since mixed strategy equilibria are pure equilibria of a game with an enlarged strategy space, there is a corollary for the mixed equilibrium case, but it uses the stronger *mixed-best-reply closure condition*. We state that condition as follows: for every mixed strategy profile $\delta_{-n} \in \times_{j \neq n} \Delta(\hat{X}_j)$, there exists $x_n \in \hat{X}_n$ such that for all $x_n' \in X_n$. $\pi_n(x_n, \hat{x}_{-n}) \geq \pi_n(x_n', \hat{x}_{-n})$.

Theorem 2. Suppose $\hat{X}$ has the mixed-best-reply closure property in (N, X, π). Then, a profile $\delta \in \times_j \Delta(\hat{X}_j)$ is a mixed Nash equilibrium of $(N, \hat{X}, \pi)$ *if and only if* it is also a mixed Nash equilibrium of (N, X, π).

III. Application to Search Auctions

For this section, we follow the earlier literature by treating bids as prices per impression rather than prices per click. As we have already described, this conversion is straightforward when search terms are interpreted narrowly; it does not affect the strategic analysis in that case.

Simplified Search Auctions Are Desirable

Suppose that bidder *i*'s value of an ad in position *n* is denoted v_{in}. Each advertiser is permitted to acquire only one ad position, so the vector v_i completely describes the bidder's values for the possible positions it might acquire. We make the standard normalization that a bidder who gets no ad has a zero payoff. Let us initially suppose that there is a small cost ε of submitting a positive bid for each position. In this model, there is no best reply to any pure strategy profile that entails a positive losing bid, so in particular the usual dominant strategy analysis for the Vickrey auction fails. That analysis does, however, have a useful counterpart in the model with costly bidding: if bidder *i* submits a positive bid $v_{in} \neq b_{in} > 0$ for just one position, then that bid is weakly dominated by $b_{in} = v_{in}$. By inspection, if bidders bid only for the items that would be assigned to them in an efficient allocation, then the corresponding singleton bids $b_{in} = v_{in}$ describe a Nash equilibrium. Summarizing:

Theorem 3. In any pure strategy equilibrium of the Vickrey auction game with costly bidding, the seller's revenue is zero. If the equilibrium bids are undominated, then the winner *i* of position *n* bids $b_{in} = v_{in}$ for that position. There is a zero-revenue undominated equilibrium in which the items are assigned efficiently. This efficient zero-revenue equilibrium bid profile is also a (dominated) pure Nash equilibrium when the bid cost is zero.

The Vickrey auction thus has undesirable Nash equilibrium properties when there is even an arbitrarily small cost of reporting bids. To make an analogous statement for a series of second price auction, we let the vector $b_i = (b_{i1}, ..., b_{iN})$ denote the bids that advertiser *i* is prepared to make for each of the *N* positions. To keep notation simple, let us permute the bidder indexes so that bidder 1 is the bidder who wins the first position, bidder 2 the second, and so on. Let $L_n = \max_{j>n} b_{jn}$ denote the second highest ("losing") bid for position *n*. In the sequence of second-price auctions, this is the price paid by bidder *n* to acquire ad position *n*. If bidder *n* makes K_n positive bids, then its payoff is $v_{nn} - L_n - \varepsilon K_n$.

Theorem 4. In any pure strategy equilibrium of the sequence of second price auctions with costly bidding, the seller's revenue is zero. If the equilibrium bids are undominated, then the winner *i* of position *n* bids $b_{in} = v_{in}$ for that position. There is a zero-revenue undominated equilibrium in which the items are assigned efficiently. This same bid profile is also a pure Nash equilibrium when the bid cost is zero.

In both the Vickrey auction and the sequence of second price auctions, the revenue result reverses when the strategy sets are simplified.

For the Vickrey auction, suppose we follow the earlier papers in assuming that bidder values per click do not depend on the ad position and that the click rate on an ad in position *n* is some fixed from α_n of the rate in position 1, where $1=\alpha_1 > ... > \alpha_N > 0$. Then, $v_i = v_{i1}(1,\alpha_2,...,\alpha_N)$; the bidder's value space is one-dimensional. The auctioneer needs only to ask each bidder for a bid b_{i1} for the first position. Since the auctioneer can observe α, it can compute the Vickrey prices for each bidder and position. In the resulting game, if there are positive bidding costs, any bid $v_{i1} \neq b_{i1} > 0$ is weakly dominated by the bid $b_{i1} = v_{i1}$ .In an undominated pure equilibrium, each of the bidders with the *N* highest values will prefer to make positive bids and the other bidders will prefer to bid zero. Position *N* will have a price of zero, but the price of any position $n < N$ is at least $(\alpha_n - \alpha_N)b_{N1} > 0$, since the opportunity cost of position *n* is not less than the gain from reassigning bidder *N* to that more valuable position.

Theorem 5. With $N > 1$ positions for sale, at least *N* bidders, and zero or small positive bidding costs, there is no zero-revenue equilibrium of the simplified Vickrey auction. At any pure equilibrium, the price paid for position *N* will be zero, but all other prices will be strictly positive.

A similar analysis applies to using single bids for a sequence of second-price auction. This is precisely the generalized second-price auction.

Theorem 6. With *N*>1 positions for sale and zero or small positive bidding costs, there is no zero-revenue equilibrium of the generalized second price auction. The price paid for position *N* will be zero, but all other prices will be strictly positive.

Only the cases with zero bidding costs are formally applications of the Simplification Theorem. For those cases, the zero-revenue Nash equilibria are eliminated by simplifying the strategy set for the Vickrey auction or the series of second-price auctions, but certain positive revenue equilibria remain. We have included positive bidding costs in this analysis because they select certain interesting equilibria and because they are an integral part of the reason for making simplifications, providing a bridge connecting the theories of sponsored search and package bidding.

The One-Dimensional Vickrey Pricing Rule is Undesirable

We have just seen that, in a particular model, a simplification that enables the auctioneer to implement Vickrey pricing from one-dimensional bids. If Vickrey pricing is both implementable and desirable, why does the search auctioneer not do that? Does to the generalized second-price auction have a heretofore unrecognized advantage?

The answer offered here uses the fact that the preceding analysis incorporates an unjustified assumption, namely, that the value of clicks is independent of the position of the ad. To explore an alternative, we introduce heterogeneity among searchers, supposing that there are two types. Searchers of one type (“potential buyers) are looking for a product to buy while those of the other (“curious searchers”) are merely looking for information. The ratio of curious searchers to potential buyers is denoted by λ.

In the prior literature, it is supposed that a searcher’s click rate on an ad is determined by multiplying the ad’s “clickability” times the click rate for the position. Here, we assume the same. For potential buyers, the relative click rate on an ad in position *n* is α_n; for curious others, it is β_n. We assume that $\alpha_1 > ... > \alpha_N > 0$ and $\beta_1 > ... > \beta_N > 0$, but we do not assume that the two series are proportional. For example, if the attention of curious searchers flags more quickly than that of potential buyers, then the sequence β_n / α_n would be decreasing.

We assume that only clicks by potential buyers are valuable to advertisers, so the value of an ad in position *n* is $v_i \alpha_n$. A bidder can learn this positional value over time by observing its sales from ads in position *n*. The formulation $v_i \alpha_n$ for the matching value implies that assortative matching is efficient, that is, the advertiser with the highest value v_i should be shown in first position, and so on for the other positions. It simplifies the exposition to label the bidders so that $v_1 > ... > v_M$ and to assume that there are weakly more positions than bidders $M \geq N$. Then, at the efficient allocation, position *n* is assigned to bidder *n*.

It has long been known that market clearing prices exist for a class of matching problems including the one described and further that there is a unique minimal market clearing price vector p which can be computed using linear programming (Koopmans and Beckmann (1957)). The minimum equilibrium price p_n is the shadow price of an additional impression in position *n*. It follows that p_n is the opportunity cost of the ad placed in position *n* by bidder *n*, so it is also the Vickrey price paid by bidder *n* to acquire that position.

Competitive equilibrium prices satisfy constraints that bidder *n* prefers position *n* to position *n*–1, that is, $v_n \alpha_n - p_n \geq v_n \alpha_{n-1} - p_{n-1}$ and, as is familiar from mechanism design analyses, the single crossing structure of preferences assumed here ensures that these hold as equalities at the minimum competitive equilibrium. Treating

$\alpha_{N+1} = 0 = p_{N+1}$, it follows that the Vickrey prices are $p_n = \sum_{k=n}^{N} (p_k - p_{k+1}) = \sum_{k=n}^{N} \left(v_{k+1}(\alpha_k - \alpha_{k+1})\right)$, which is the formula for such prices reported by Edelman, Ostrovsky, and Schwartz (2007).

The click rate for position *n* is $\alpha_n + \lambda\beta_n$. Although this rate decreases with *n*, it would be a rare coincidence for it to decrease in direct proportion to the value of an ad. Since the search company observes clicks but not sales, it varies bids in proportion to clicks but not in proportion to value. If bidder *i* names a price of b_{i1} for position 1 in a simplified auction, then the auctioneer can impute a bid for position *n* as $b_{i1}\gamma_n$, where $\gamma_n = (\alpha_n + \lambda\beta_n)/(\alpha_1 + \lambda\beta_1)$ is the relative click rate for position *n*, but the auctioneer *cannot* generally infer Vickrey prices from these bids and its other information.

Is the efficient assignment with the Vickrey price vector *p* is the outcome of Nash equilibrium in the generalized second-price auction? If it is, then it must be that the highest bid is made by bidder 1, the second highest by bidder 2, and so on, and that the highest losing bidder for each position bids the Vickrey price for that position. Thus, for each bidder *n* for $n = 2,..., N+1$, it is necessary that the equilibrium bids are $b_{n1} = p_{n-1} / \gamma_{n-1}$. The other bids are not uniquely determined, but we may specify that bidder 1 bids $b_{11} = \alpha_1 v_1$ and that bidders with indexes *N*+1 and larger bid $b_{n1} = \alpha_N v_n / \gamma_N$.

Theorem 7. For the two searcher-type model of this section, there is a pure Nash equilibrium of the generalized second-price auction in which the assignment is efficient and prices paid by the winning bidders are the Vickrey prices *p if and only if* the corresponding price-per-click sequence $\{p_n / \gamma_n\}_{n=1}^{N}$ is decreasing.

Proof. If the Vickrey-price-per-click sequence p_n / γ_n is not decreasing, then the bidders are not ranked in the correct order for an efficient assignment. (For example, if $p_3 / \gamma_3 < p_4 / \gamma_4$, then bidder 4 bids less than bidder 5 and the resulting assignment is inefficient.)

Suppose that $\{p_n / \gamma_n\}_{n=1}^{N}$ is decreasing and fix any bidder *n*. Recall that the Vickrey prices are competitive equilibrium prices so no bidder wishes to deviate to purchase a different position at prices *p*. If bidder *n* raises its bid to win a higher position, say position $k < n$, then the price it must pay is determined by the k^{th} highest bidder, so it is $\gamma_k(p_{k-1} / \gamma_{k-1}) > \gamma_k(p_k / \gamma_k) = p_k$, so that deviation is unprofitable. If bidder *n* reduces its bid to win a lower position $k > n$, then the price it must pay is precisely p_k and the deviation is again unprofitable. ♦

Previous literature establishes that the desired equilibrium exists when $\lambda = 0$ or more generally when the vector γ is proportional to the vector α, that is, when the seller's estimate of relative values is not too far off. When the values v_i of the various bidders are very close, then this condition is almost necessary, so the generalized second-price auction does not work well. When the values variation is larger, this constraint is more relaxed.

In any series of second-price auctions in which advertisers other than *j* were obliged to use one-dimensional strategies, suppose that a best reply by *j* wins some position *n*. The price *j* pays in that case is determined by the n^{th} highest opposing bid. It can obtain the same position at the same price with a one-dimensional bid that is the n^{th} highest such bid. Therefore, we have proved the following.

Theorem 8. The generalized second-price auction satisfies the best-reply closure property.

Using Theorem 8, we can apply the Simplification Theorem. The pure Nash equilibria of the generalized second-price auction are also equilibria of any sequence of second-price auctions with richer strategy sets. As we have seen, the full set of such equilibria for the richer game include ones with zero revenues. This identifies an advantage of the generalized second-price auction as it is actually conducted for sponsored search applications.

The analysis reported in this section was formulated for application to online search, but similar analyses can be developed for other Internet advertising auctions. The reason conflation is valuable is that advertising targets can be too highly differentiated. For example, a Palo Alto mortgage lender might be prepared to bid high to target a refinancing online ad to "males aged 35-54, homeowners in Palo Alto, CA, with good credit scores whose navigation behavior displays interest in home improvement or mortgage refinance and who are not currently visiting a sex, gambling or gaming site." Detailed targeting can be valuable because it improves the matching of ads to users, but too narrow targeting can result in little competition and low revenues for many ad placement opportunities. From this perspective, sponsored search is just one example in which a simplified auction that conflates distinguishable ad opportunities both supports high quality matching and generates significant equilibrium revenues.

IV. Application to Package Auctions

In contrast to the assumption made in much economic theorizing that auctions are conducted for a single item, many auctions take place in settings where multiple items are being sold and the sales interact. This relationship can emerge from budget constraints that prevent independent bidding on separate items. It can also emerge when the goods enter the buyer's production or utility function as substitutes or complements. Although such interactions are very common, *package auctions*, in which bidders can name prices for the packages of lots or items they wish to buy, are only infrequently used.[6] More often, items/lots/tranches are sold sequentially or in simultaneous sealed bids. The use of these alternative arrangements calls for explanation.

It seems intuitively clear that these one-item-at-a-time auctions are simpler than package auctions, although the rubric "simple" is an ambiguous one. One important meaning that has received some attention is that computation is much easier for single item auctions than for package/combinatorial auctions. A second simplicity notion, which we have emphasized in this paper, is that bids are restricted so that bidders are called upon to make fewer bids.[7]

Many common single item auctions are simplified package auctions according to our definition. For example, a simultaneous second-price auction for *N* items is a simplification of a standard Vickrey package auction for *N* items in which bidders are allowed to make only bids that express values of packages as the sum of the values of their constituent items. Also, a simultaneous first-price auction is the simplification of a Bernheim-Whinston menu auction with the same bid restriction.

Many more complex package auctions impose restrictions on bids that qualify them as simplified package mechanisms in the sense introduced here. For example, the

[6] A recent book by Cramton, Shoham, and Steinberg (2005) reports a snapshot of the growing literature on package auctions, including reports of applications. Milgrom (2004) describes additional applications.
[7] This type of simplicity is relevant for reporting and computation, too, since the amounts of reporting and computing time are functions of the amount of data.

City of London procures bus services using a package auction which requires bidders to submit a price meeting the reserve for each named route while allowing discounts to be offered for combinations of routes (Cantillon and Pesendorfer (2005)).

Below, we limit attention to simplifications of core-selecting package auctions. The underlying direct mechanisms are ones that always select an allocation in the core determined by reported values. Among these mechanisms are the menu auctions studied by Bernheim and Whinston (1986). Those authors showed that for every *bidder-optimal* allocation (meaning a core allocation that is not Pareto dominated for the bidders by any other core allocation), there is a coalition-proof equilibrium of the menu auction which selects that allocation. If π is the corresponding bidder-optimal core imputation, then the equilibrium strategy profile has each bidder *j* report that each non-empty package *S* has value $\max(v_j(S)-\pi_j,0)$, where v_j is the bidder's actual value function for packages. We denote this report by $v_j-\pi_j$.

Day and Milgrom (2007) show that precisely these same profiles of *profit-target strategies* $v_j-\pi_j$ are Nash equilibria of *every* core selecting auction mechanism. They also show that for every core-selecting auction and every strategy profile of the other bidders, bidder *j* has a best reply of the form v_j-k for some $k\geq 0$. The theory we develop below applies to this whole set of auction mechanisms.

Consider a simplified core-selecting auction in which bidders are restricted to report values in a set *V*. With a set of items *N* for sale, a typical element $v\in V$ is a function $v:2^N\to\mathbb{R}_+$ with the property that $v(\varnothing)=0$. For $k>0$, let $v-k$ denote the value function which assigns to any non-empty package *S* the value $v(S)-k$.

Definition. The set of values V is *closed under fixed costs* if for all $k > 0$, $v \in V \Rightarrow v - k \in V$.

A direct application of Theorem 2 of Day and Milgrom (2007) yields the following result.

Theorem 9. Let Γ_V be a simplified core-selecting auction with reports restricted to lie in the set V. Suppose that V is closed under fixed costs and that actual bidder values lie in the set V. Then, Γ_V has the best-reply closure property and the profit-target equilibrium strategy profiles identified above for the full mechanism are also equilibrium of the simplified mechanism.

Theorem 9 identifies a class of simplified mechanisms for package bidding. For example, V might be the set of values expressed as the sum of item values, minus a constant: $v \in V \Leftrightarrow \left(\exists \alpha \in \mathbb{R}_+^N, k \in \mathbb{R}_+\right)(\forall S \neq \varnothing) v(S) = \sum_{n \in S} \alpha_n - k$. Elements of V could express values of collections of items when there is a fixed cost of shipping or a shared facility that must be built to use the items. Simplified core-selecting mechanisms using this V can be dubbed *fixed cost package auctions*.

Among the important features of the fixed cost package auctions is that they eliminate many (but not all) coordination failure equilibria. For example, suppose that $N = \{1, 2, 3\}$ and that there are three bidders. Suppose that bidder 1 values only item 1 and has a value of 10; bidder 2 values only items 2 and 3 with values of 10 each and fixed costs of 10, and that bidder 3 values the items at 5 each, with no fixed cost. Among the Nash equilibria of the full menu auction is one at which bidder 3 wins all the items, bidding 15 for the whole set and making no other bids, while bidders 1 and 2 each bid 10 for the whole set, making no other bids. There is no corresponding equilibrium of the simplified game. If bidders 1 and 2 play only undominated strategies and bid their full values for the package of the whole, then the only corresponding equilibrium outcome entails an efficient allocation. This illustrates the Simplification Theorem, according to which the narrower strategy set can eliminate equilibria but cannot introduce additional equilibria.

Two other important advantages of the fixed cost package auction design are the low dimensionality of the reports required from bidders and the fact that for any fixed number of bidders, computation time rises only linearly in the number of items for sale.

Affine Approximation Mechanisms

Here we propose a simplified mechanism that incorporates the fixed cost package auctions while preserving all of its advantages and also extends a design created by the author to sell the generating assets of an electric utility company. In the asset sale application, two kinds of bidders were expected to participate in the auction—ones that wanted to buy all or nearly all of the generating portfolio and others that wanted to buy only specific very small parts of the portfolio. For example, the company's partners in ownership of some electric generating facilities might want to buy the selling company's share in order to avoid being saddled with unfamiliar new partners and counterparties to certain contracts might want to buy back their commitments. The suggested design involved two stages[8] of which the second involved a package auction in which bidders for the whole portfolio of assets would be required to specify decrements to be applied to their bid for the whole portfolio if some of the individual pieces were sold to others.

[8] The first stage involved indicative bids to identify qualified bidders and to determine which assets would be open for individual bidding.

Partners and counterparties bidders could bid for the individual pieces for which they were qualified.

Generally, we define the *affine approximation mechanisms* to be simplified core-selecting auctions in which a bid (T, β, α, r) comprises a package *T*, an offer β for that package, individual item prices $\alpha \in \mathbb{R}_{+}^{N}$, and a radius of approximation $r \geq 1$. The bids can be used to impute a value function for non-empty packages for the core-determining engine according to the formula

$$v(S) = \begin{cases} \beta + \sum_{n \in S-T} \alpha_n - \sum_{n \in T-S} \alpha_n & \text{if } \max\left(|S-T|, |T-S|\right) \leq r \\ 0 & \text{otherwise} \end{cases}$$

where $|S-T|$ and $|T-S|$ are the numbers of elements in $S-T$ and $T-S$, respectively. Thus, the tuple (T, β, α, r) is understood to specify an offer of β for package *T* and adjustments for packages that are similar to *T*. Adding and/or deleting up to *r* items from the package *T* alters the bid by adding and subtracting the corresponding item prices. Adding and/or subtracting more than *r* items results in a zero bid (though it is should be evident from the logic that other specifications besides zero could also work here). The asset sale described above is a further simplification that restricts the sets *T* and the radius *r*. We denote by $\hat{V}$ the set of values that can be reported without any restrictions on *T* or *r*.

Even with restrictions on *T* and *r*, the set of values is plainly closed under fixed costs so when values are actually of this class, it has the best-reply closure property. It is unlikely, however, that values will often lie in that class, so we are led to ask: what happens when the actual package values do not lie in the set $\hat{V}$?

Theorem 10. Let $\Gamma_{\hat{V}}$ be the simplification of a core-selecting auction with reported valuations restricted to lie in $\hat{V}$. Then, regardless of the bidders' actual valuations, this mechanism has the best-reply closure property. (The same is true even when *r* is restricted, but not when *T* is restricted.)

Ignoring the role of ties, the proof can be put briefly in words as follows. Fix some bidder *j* and strategies in $\hat{V}$ for the other bidders. Suppose there is some best reply report by *j* that wins some non-empty package *T* at price p_T. Let $\alpha_j \in \mathbb{R}_+^N$ be any vector with the property that for all $n \in T$, $\alpha_{jn} > \alpha_{j'n}$ for all other bidders $j' \neq j$ and for $n \notin T$, $\alpha_{jn} = 0$. Since the auction selects core allocations with respect to the reports, the allocation selected by the original best-reply has a higher total value than any allocation that excludes *j*. So, *j* must still be a winner with the proposed bid. By construction, the value-maximizing outcome when *j* is included assigns package *T* to *j*. Also, since the core requires individual rationality, the price that *j* pays cannot exceed p_T. Hence, the proposed bid in $\hat{V}$ is a best reply for *j* to the given opposing strategy profile. This bid is a best-reply for any value of *r*, so restrictions on *r* do not change the conclusion.

One interesting aspect of the affine approximation mechanisms is that they use something resembling prices to guide the allocation of items among the winning bidders. The idea of using item prices to guide package allocation has been repeatedly proposed in recent years. It is incorporated in the FCC's current package bidding algorithm and in the dynamic algorithms suggested by Porter, Rassenti, Roopnarine, and Smith (2003) and by Ausubel, Cramton, and Milgrom (2005). All of these mechanisms, however, impose upon prices the burden of guiding both the winner determination problem—which bidders should be in the winning set—and allocations of items among the winners.

The approximation mechanisms do not work that way: they attempt to utilize item prices to allocate goods among the winners but not by themselves to determine the set of winning bidders. The FCC's experiments with its package auction design shows that these item prices are highly unstable during the course of an ascending auction, increasing and decreasing by large amounts over time. In the perspective taken here, the proper item prices to guide the allocation of items among winners depends on the set of winners. If these are changing during an ascending package auction, then sharp swings in the supporting prices are to be expected.

The affine approximation mechanism with no restrictions on T or r may be useful in some settings with small number of items, but as the number of items grows large, they may admit too many coordination failure outcomes in which the number of packages implicitly bid by each bidder is too small. For some applications, one might require $r = N$, so that all bids are based on a single affine approximation of each bidder's value function. Such a mechanism makes computation easy and transparent and reduces size of the bid/report from something that is exponential in N to something that is linear in N. More generally, restricting T and/or requiring a wide radius of approximation r or using a better approximation than the affine one may be workable simplifications for some applications.

Small Bid Costs

The idea that bids costs are significant in package auctions even with relatively few items seems compelling—with $N = 10$ items, there are $2^N - 1 = 1023$ non-empty packages. Nevertheless, the best way to introduce these costs into the analysis is not obvious. One particularly simple alternative is to assume that costs are zero for simplifications that make the number of reports rise only linearly in N and the cost is otherwise prohibitive. By this standard, the affine approximation auctions described above are zero cost mechanisms, while full menu auctions are prohibitively costly. If the bid reductions are left to the bidders, there are many equilibria involving coordination failures, where packages in the efficient allocation receive no bid at all.

Another approach to bidding costs, more consistent with the treatment of sponsored search auctions above, is to assume that there is some small cost $c > 0$ of reporting each number. The difficulties this poses for equilibrium analysis are most simply illustrated by considering the case of a single item for sale: N=1. Suppose there are two bidders: a high value bidder 1 with value v_1 and a low value bidder 2 with value v_2. In the second-price auction in this case, the only full-information equilibrium has bidder 1 bid v_1 while bidder 2 bids zero, so the seller's revenue is zero. The first-price auction has no full information pure equilibrium when bid costs are small and positive. For if there were such an equilibrium and the equilibrium price were less than v_2, then both bidders 1 and 2 would enter, leading to a higher price than v_2. Alternatively, if the equilibrium price were v_2 or higher, then only bidder 1 would enter, so the price would be zero. It seems sensible for this case to model small bid costs by focusing on a pure price that is a limit of mixed strategy equilibria with random participation by bidder 2. This limiting price must be v_2, for if the bidder 2 randomizes about entry, its equilibrium profit must be zero, so the probability that a bid of $v_2 - c - k$ wins can be no more than c/k.

This analysis points to a revenue advantage to using first-price auctions rather than second-price auctions when bid costs are positive but small. Day and Milgrom (2007) reach an opposite conclusion using a different idea, namely, that it is cheaper to bid straightforwardly than to base each bid on a strategic calculation, so that the cost of bidding is less in a second-price auction. This may also encourage more entry. Neither of these effects appears in our full-information equilibrium analysis, but that is an outcome of the particular and extreme assumptions required for such an analysis. Our model is not well suited to assess the comparative importance of these competing effects, but it does succeed in highlighting a new and potentially significant effect.

V. Conclusion

That simplicity is desirable is hardly controversial among auction designers, but there has been little discussion about what "simplicity" may mean or what advantages it may convey. Here, we begin to tackle that question by defining a *simplified mechanism* to be a direct mechanism but with a restricted *bidding language*, that is, a restriction on the set of permissible reports or bids. We have shown that simultaneous independent first- or second-price auctions are simplified package auction mechanisms in precisely this sense.

One type of simplification found frequently in practice works by restricting the set of bids to force the same bid to apply to two or more distinct items. Such bids may be dubbed *conflations*. The sponsored search auction requires the use of conflations because the same price per click must be offered for all ads regardless of their positions on the search page. Treasury bills, which differ only in their serial numbers, are such an obvious candidate to be sold using conflations—bids express only the total face value of the bills to be purchased (without regard to the serial numbers) and the price offered—that one might not even notice that this is a simplification. Yet, the restriction that bids cannot depend on the serial number conveys the same advantages as the conflations required in of sponsored search, that is, it eliminates low revenue equilibria (including both pure and mixed equilibria).[9] Conflations are also used in certain electrical power auctions, when "zones" are established within which power or capacity is treated as a single undifferentiated commodity. This may be done even though substitution among power sources or sinks within a zone is imperfect.

One implication of all these examples is that conflations can increase competition for each of several goods by forcing bids on one to be bids on all. Yet not all conflations work equally well. In daily electrical power markets, the system operator typically acquires both base load generation capacity and load-following *regulation*—the latter is capacity that can produce power that follows the "load" (the power demanded) as it fluctuates from minute to minute. In California, bids to supply regulation were for a period treated as distinct from bids for base load capacity—a failure to conflate properly. In this case, the proper conflation is asymmetric: a bid for regulation should also count as a bid for base load capacity. The old system sometimes deprived the market of actually available base-load supply resulting in unnecessarily high prices.[10] This California case points both to the tendency of practitioners to adopt simplified auction designs and to the importance of choosing the right conflation.

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[9] In T-bill auctions, the bills are actually perfect substitutes, so the auction restricted to conflations satisfies that best-reply closure property holds even in *mixed* strategies. To illustrate an advantage of conflations when bidding is costly, suppose there are N bills and N+1 bidders, that each bill is worth 1 to each bidder, and that each bidder can costlessly bid for one bill but incurs a cost to bid for two or more. Then, the unique Nash strategy equilibrium of the simplified first-price mechanism with a zero minimum bid has revenue of N, but no equilibrium of the auction for N individual items has revenue greater than 1.

[10] To illustrate how this can happen at equilibrium, imagine that demand fluctuates between 1 and 2 units and that there are three suppliers, each capable of supplying one unit and two capable of supplying regulation services by following the load fluctuations. If the two markets for base load and regulation are run separately and simultaneously, then there is a necessarily a single bidder in one of the markets.

In our theoretical account, simplification can have several advantages. First, in multi-product auctions, simplification can save costs by obviating the need to bid separately for all the possible alternatives. Second, in the same setting, simplification can improve performance because, without a simplified set of bids, bidders may make too few bids, damaging efficiency and reducing revenues. For sponsored search auctions with positive bid costs and without simplification, we found that every full-information equilibrium entails zero seller revenues (for both the Vickrey design and the series of second-price auctions); in contrast, there are no zero-revenue equilibria in suitably simplified versions of these auctions. Third, even when bidding costs are zero, the unsimplified direct mechanism can have multiple Nash equilibria, some of which entail undesired outcomes. The Simplification Theorem applies to this zero-cost case, asserting that a simplification satisfying the best-reply closure property which eliminates some equilibria nevertheless introduces no new equilibria. So, the full set of pure equilibria of a simplified auction can have advantages compared to the equilibrium set of the original direct mechanism.

Our theoretical account is based on a formal model that captures some, but hardly all, of the important aspects of simplified designs. It does not account for learning, which one might conjecture is faster and more precise in a simpler mechanism. It does not accommodate the confusion that is created by complex mechanisms. It omits the resistance of bidders to participating in too complex a mechanism. Any of these features could be quite important.

Simplification is an essential aspect of practical mechanism design.

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