# Performance and Turnover in a Stochastic Partnership

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January 13, 2010

**Abstract**

This paper characterizes the welfare-maximizing equilibrium performance and duration of stochastic partnerships, in an economy in which partners choose each period costly observable efforts, voluntary wages, and whether to leave their current relationship to form a new partnership. Welfare comparative statics are also provided: higher states are associated with higher joint payoffs and, in the special case of an exogenous stochastic process, with both higher joint stage-game and joint continuation payoffs as well as longer-lasting relationships. Unlike in non-stochastic repeated games with re-matching, (i) equilibrium social welfare is maximized by renegotiation-proof play within each partnership and (ii) maximal equilibrium welfare is strictly decreasing in the cost of partnership formation.

## 1 Introduction

Players in an ongoing interaction often face uncertainty regarding the fundamentals of their relationship. For example, an employer may be unsure about whether his worker will have an

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incentive in the future to accept an outside offer. Or, firms engaged in a joint venture may be unsure about future payoffs within their partnership. Such uncertainty can make it difficult to sign complete formal contracts, especially if what might change in the relationship is difficult to communicate to an outside party. At the same time, a long-lasting (or "stable") relationship is crucial for the effective provision of informal incentives. If shocks to the productivity of a partnership may cause it to end or be less productive in the near future, players will have less incentive to work today, reducing relational gains and potentially hastening the partnership's demise in a vicious cycle.

Partnerships in which players face an uncertain future can generate rich dynamics. For example, one of General Motors' most important joint ventures is Shanghai GM, a 50-50 partnership with Shanghai Automotive Industry Corp (SAIC) formed in 1997 that has grown into a fully integrated operation that moves over one million cars per year. This venture allows GM to build cars in China, while providing SAIC with valuable expertise. In fact, GM has aggressively transferred vehicle development know-how to its Chinese partner, as recently as 2008 with the complete re-design of the Buick Regal for the Chinese market, despite some mimicking of its past designs in SAIC vehicles. Yet, when announcing plans in 2007 to build a hybrid-engine research center in Shanghai, GM chose not to work with SAIC, even as the Chinese government announced seemingly strict rules requiring the manufacture of hybrid engines on Chinese soil (Bradsher (2007)).

As in many uncertain relationships, this partnership's future is subject to a variety of risks, some within the players' control (will GM continue to refuse to share valuable intellectual property, will SAIC continue to copy-cat GM designs?) and others less so. Indeed, the future of GM Shanghai may well hinge on developments in another stochastic relationship, that between the Chinese government and foreign auto-makers. For example, should China stop foreign firms from exporting earnings, or allow them to build hybrid cars without a Chinese partner, GM would have much less incentive to share its technology.

This paper develops a theory of *endogenous stability and performance* in a perfect-information model of stochastic partnerships. Each period, two partners simultaneously decide how much effort to exert after observing a payoff-relevant state. "Effort" can be interpreted broadly, e.g. to include costly relationship-specific investments. After observing efforts, the partners then simultaneously decide whether to quit the relationship. The partnership ends if either player quits, in which case each player receives an outside option. (Players can also make voluntary wage transfers at any time during the game.)

The model imposes only a few substantive restrictions on stage-game payoffs and on the stochastic state of the partnership. Notably, stage-game payoffs are assumed to satisfy increasing differences in players' efforts and the state, while the stochastic process is assumed to satisfy a positive serial auto-correlation property that higher past states make higher future states more likely in the sense of first-order stochastic dominance. However, no substantive restrictions are placed on how efforts control the stochastic process. This allows for a rich set of potential applications from labor to macroeconomics and organizational economics, in which greater effort grows, depletes, or has a non-monotone effect on a payoff-relevant relational stock. For example, in a labor context, one could interpret the worker's (multi-dimensional) effort as including hours worked as well as investments in firm-specific human capital. The assumptions are sufficiently weak that the existing literature on comparative statics in stochastic games does not apply. (See the literature discussion below.)

The analysis is divided in two parts. In the first part (Section 4), I derive a subgame-perfect equilibrium (SPE) that maximizes players' joint welfare among all SPE, *taking as given* the players' outside options (Theorem 1). Joint payoff in this "optimal equilibrium" is non-decreasing in the state (Theorem 2), but higher states need *not* in general be associated with higher joint stage-game payoff or higher joint continuation payoff. Consequently, players in higher states may or may not exert more effort, may or may not exit with lower probability, etc. However, more comparative statics are available in the special case in which players' efforts do not control future payoff possibilities. In this case, partnerships in higher states will enjoy (weakly) higher stage-game payoffs, higher continuation payoffs, and later stopping times in the sense of first-order stochastic dominance (Theorem 3).

In the second part (Section 5), this partnership game is embedded within a "partnership economy" with immediate, anonymous re-matching after partnership dissolution. If some player's partnership ends at time $t$ in this economy, whether because he left, his partner left, or his partner died, he is automatically re-matched with a new partner to begin at time $t+1$. This new partnership is assumed to be a "fresh start", in the sense that

(i) the stochastic processes driving stage-game payoffs are iid across partnerships and (ii) players know nothing about their current partner's history before their partnership began, including his age, number of past partnerships, etc.[1] Expected payoffs in a new partnership generate outside options for each player should his current partnership end. The analysis endogenizes the maximal joint outside option that can be supported in any equilibrium of the partnership economy, thereby closing the model in a general equilibrium framework (Theorem 4). Further, given an exogenous inflow and outflow of births and deaths, I characterize the steady-state distribution of histories among active partnerships in this social welfare-maximizing equilibrium.

As long as re-matching is costless and anonymous, cooperation within any given partnership requires those who leave a relationship to face some *endogenous* cost of being re-matched. For this reason, in non-stochastic repeated games with re-matching, maximizing social welfare requires that partners "burn money" or otherwise fail to immediately achieve the full potential equilibrium benefits of their relationship (see e.g. Carmichael and MacLeod (1997) and Kranton (1996)). Since such costs are avoidable, social welfare-maximizing equilibrium play necessarily fails to be renegotiation-proof. By contrast, under relatively mild assumptions on the stochastic process driving payoffs, equilibrium social welfare here is maximized when players within each partnership maximize their equilibrium joint payoff given their endogenous outside options. Intuitively, the reason is that each player fears that his next partner may be a "bad fit" and hence treats his current partner well without the need to commit to burn money. This difference has important economic implications. For example, increasing the cost of partnership *formation* strictly decreases equilibrium social welfare when payoffs are stochastic, but not when payoffs are non-random (Claim 4).

Since players immediately leave all but the best matches, there is a *"honeymoon effect"* to partnership formation. Namely, partnerships that persist at all are likely to last a relatively long time and to be highly productive at first. Exit is triggered when the state of a partnership falls below a threshold-surface in the state-space. Consequently, partnerships that have lasted a long time tend to be those that have received mostly positive shocks

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[1] If historical variables such as age could be observed, then economy-wide welfare might be enhanced in "old-maid equilibria" in which players who are not newly-born are shunned.

that made the partnership more stable. This *"survivorship bias"* is consistent with a broad empirical finding that, from employment (Topel and Ward (1992)) to marriage (Stevenson and Wolfers (2007)) and organizational survival (Levinthal (1991)), partnerships that have lasted a long time are less likely to end in the near future.

There is a range of states ("hard times") in which partners exert little or no effort in the optimal equilibrium but elect to remain together despite this failure to cooperate. Players endure such hard times, rather than exiting, because of the *option value* associated with waiting to exit. However, this option value does not only arise as usual from exogenous variation in the productivity of the partnership itself. The option to exit later becomes more valuable, in equilibrium, because of the endogenous variability of players' behavior.

The rest of the paper is organized as follows. The introduction continues with discussion of some related literature. Section 2 describes the model while Section 3 provides a simple example. Section 4 characterizes the joint welfare-maximizing SPE in a partnership for any outside options. Section 5 then extends the analysis to a setting with re-matching after partnership dissolution, thereby endogenizing each player's outside option. Section 4.1 develops more comparative statics in the special case of an exogenous stochastic process, while Section 6 provides concluding remarks. Most proofs are in an Appendix.

**Related literature.**

This paper synthesizes elements from the literatures on productivity shocks (e.g. Jovanovic (1979a)), relational contracts (e.g. Levin (2003)), and repeated games with re-matching (e.g. Kranton (1996)), in a rich but tractable stochastic framework.

Jovanovic (1979a) considers a model in which a worker learns over time about the productivity of the match with his present firm and quits as soon as he becomes sufficiently pessimistic about the match. Consequently, workers who have remained longer at the same firm are less likely to leave and more likely to be more productive.[2] The key difference here is that partners face a two-sided incentives problem as well as a learning problem. Whereas the

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[2] Also closely related is Jovanovic (1982), in which each firm's growth rate and survival depends on what it learns about own productivity, and Jovanovic (1979b), in which similar effects arise as workers who choose to remain in their current job make firm-specific investments to improve the future performance of the match. See also Pissarides (1994) for a related model of on-the-job search.

worker in Jovanovic always enjoys the full gains from his current match, players here must work to enjoy those gains and choose how to distribute them through voluntary wages. Levin (2003) characterizes optimal "relational contracts" in a principal-agent context in which the agent's cost of effort is iid. Unlike Levin (2003), this paper allows for two-sided incentives and non-iid stage-game payoffs, and endogenizes players' outside options through a re-matching technology.[3]

The analysis here confirms key qualitative findings from the literatures on productivity shocks and relational contracts. For example, I show that performance *inside* the partnership decreases with the attractiveness of players' outside options. This extends a well-known finding of the relational contracts literature (see e.g. MacLeod and Malcomson (1989) and Baker, Gibbons, and Murphy (1994)) to a richer stochastic setting. Similarly, the fact that partnerships in the social welfare-maximizing equilibrium typically exhibit a honeymoon and a survivorship bias is qualitatively similar to Fichman and Levinthal (1991)'s findings about firm performance and survival when productivity follows a random walk.

On the other hand, some of these same findings are quite surprising when viewed from the perspective of repeated games with re-matching (see e.g. Kranton (1996), Datta (1996) and recently Eeckout (2006) and Fujiwara-Greve and Ohuno-Fujiwara (2009)). A key finding of this literature is that social welfare is maximized when partners *fail* to realize all potential equilibrium gains in their individual partnerships; instead, they burn money, forego profitable cooperation on the basis of payoff-irrelevant information, or "start small" with low efforts and low payoffs before transitioning to a maximally productive phase.[4] The analysis here shows that such results hinge crucially on the assumption of non-random payoffs. Under mild conditions on the stochastic process driving payoffs, welfare-maximizing equilibria of the overall partnership economy dictate renegotiation-proof play within each partnership, punctuated by intervals between partnerships in which players "date" several partners before staying with a sufficiently attractive new match.

Another counter-intuitive finding in this literature is that sufficiently small partnership

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[3] However, Levin's analysis is *not* less general, as he allows for incomplete information and imperfect monitoring of effort.

[4] In an incomplete information setting, Ghosh and Ray (1996) and Watson (1999) provide a separate, signaling rationale for starting small.

formation costs have no effect on maximal equilibrium social welfare. Players adapt to such costs simply by burning less money at the start of each relationship. By contrast, in this paper's stochastic setting, social welfare is strictly decreasing in the cost of partnership formation. Although higher partnership formation costs deter exit and thereby improve performance within each existing match, this positive effect is always overwhelmed by the combined negative effects that players must pay to find a partner and that, for this reason, they will tend to sample less and settle into long-term relationships with players who are not as good of a fit.

This paper also adds to the "dynamic games" literature in which a payoff-relevant state follows a known stochastic process.[5] For example, a key insight in Haltiwanger and Harrington (1991) and Bagwell and Staiger (1997)'s models of collusion and the business cycle, that collusion thrives at those times when the *future* state is most likely to be conducive to collusion, is helpful for interpreting this paper's results as well. However, the focus here is on how players' ability to dissolve their partnership interacts with their incentive to exert costly effort. Also, by allowing for any persistent stochastic process, my analysis encompasses both the iid case (as in Rotemberg and Saloner (1986), Ramey and Watson (1997)) and the "positively autocorrelated" case (as in Bagwell and Staiger (1997)), among others.

Like this paper, Roth (1996) shows how to construct welfare-maximizing equilibria in a dynamic partnership, using an algorithm in the spirit of Abreu, Pearce and Stacchetti (1990). Indeed, Roth's model can be viewed as a special case of mine in which, among other things, payoffs are symmetric, the state is one-dimensional and follows a simple random walk, and there is no feedback of effort on future states. Also, this paper differs by endogenizing the players' outside options via re-matching.

Recently, Chassang (2010a) and Bonatti and Horner (2010) have developed other theories of cooperation dynamics. Namely, Chassang (2010a) shows how players "build routines" in repeated games with incomplete information about payoffs, while Bonatti and Horner (2010) develop a theory of dynamic public good provision given unobserved efforts and uncertainty about the quality of the public good. In each of these papers, the underlying

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[5] A growing and less closely related literature considers dynamic games in the presence of imperfect information, e.g. Athey and Bagwell (2001) and Horner and Jamison (2007).

environment does not change over time. This paper highlights dynamics that arise when payoffs are stochastic, while abstracting from (important) issues of incomplete information and imperfect monitoring.

More tangentially related is the existing literature on “stochastic games”, especially those papers such as Amir (1996) and Curtat (1996) in which sufficient monotone structure is imposed to generate comparative statics. However, most of these papers focus on equilibria in Markov strategies, often proving uniqueness of such equilibria, whereas I consider subgame-perfect equilibria (SPE) and focus on the SPE that maximizes joint welfare among all SPE. Further, this literature imposes stronger assumptions than are needed here, in large part because they prove stronger results (such as uniqueness).

Lastly, although players have the option to exit, the literature on so-called “option games” is unrelated. In an option game, players’ payoffs depend upon who exercises a real option (e.g. exiting a market) and when they do so, and papers in this literature tend to focus on issues of strategic pre-emption or delay that arise when players prefer to be the first or last to exercise their option. See e.g. Grenadier (2002) and Chassang (2010b). By contrast, my focus is to endogenize the productivity of the match itself.

## 2 Model

Two asymmetric players in a partnership each seek to maximize the expected present value of their stream of payoffs, given shared per-period discount factor $\delta < 1$. All assumptions presented here apply throughout the analysis. Those meriting further discussion are highlighted and numbered. (Additional assumptions will be explicitly stated when needed.)

*Notational shorthand.* To improve clarity and shorten equations, I have adopted several notational conventions throughout the paper. First, random variables are capitalized while realizations are in lower case, e.g. $x_t \in \text{supp}(X_t)$. Second, variables specific to a player and time have two subscripts, e.g. $e_{it}$. Vectors of such variables for all players and *all times no later than* $t$ are bolded with one subscript, e.g. $\mathbf{e}_t = (e_0, ..., e_t)$, while those for all players *at one time* $t$ are unbolded with one subscript, e.g. $\pi_t(e_t; x_t) = (\pi_{it}(e_{it}, e_{jt}; x_t), \pi_{jt}(e_{it}, e_{jt}; x_t))$. Finally, *sums* are denoted by a summation subscript, e.g. $\pi_{\Sigma t}(e_t; x_t) = \sum_i \pi_{it}(e_t; x_t)$.

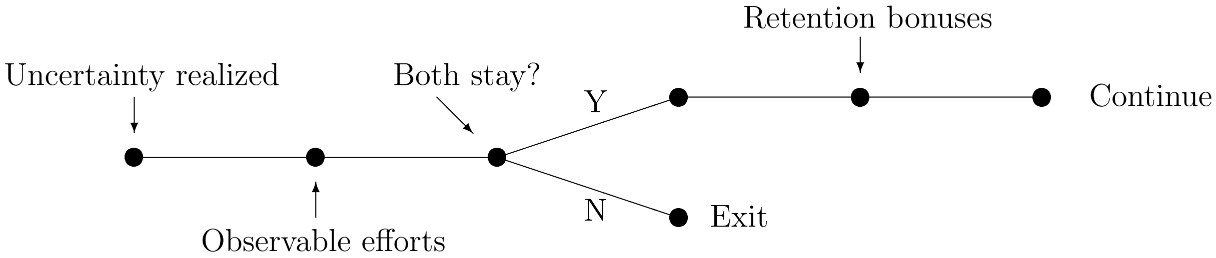

Figure 1: Timing of the partnership stage-game in period $t = 0, 1, 2, ...$

**Partnership stage-game.** Each period $t = 0, 1, 2, ...$ proceeds as follows until the partnership ends (see Figure 1). First, a payoff-relevant state $x_t \in \mathcal{X}_t = \text{supp}(X_t)$ is realized and publicly observed. $(\mathcal{X}_t, \succeq)$ is a partially ordered set. Second, each player $i$ simultaneously decides what effort $e_{it} \in \mathcal{E}_{it}$ to exert, where efforts may control the stochastic process $(X_t : t \geq 0)$.[6] Efforts are then publicly observed and each player $i$ receives "inside payoff" $\pi_{it}(e_t; x_t)$. $(\mathcal{E}_{it}, \succeq)$ is a partially ordered, complete topological space having minimal element "0".[7]

**Assumption 1** (Inside payoffs)**.** $\pi_{it}(e_t; x_t)$ is weakly decreasing in $e_{it}$, weakly increasing in $e_{jt}$, and continuous in $e_t$, $\pi_{it}(0, 0; x_t) = 0$ for all $x_t$, and $\pi_{\Sigma t}(e_t; x_t)$ is uniformly bounded above.

**Assumption 2** (Increasing differences)**.** $\pi_{it}$ satisfies weakly increasing differences in $(e_t; x_t)$. That is, $e_t^H \succeq e_t^L$ and $x_t^H \succeq x_t^L$ implies $\pi_{it}(e_t^H; x_t^H) - \pi_{it}(e_t^L; x_t^H) \geq \pi_{it}(e_t^H; x_t^L) - \pi_{it}(e_t^L; x_t^L)$.

**Assumption 3** (State)**.** The distribution of $X_t$ depends only on $(t, x_{t-1}, \mathbf{e}_{t-1})$.

Third, each player $i$ simultaneously decides whether to stay or quit the partnership. The partnership ends if either exits *and*, should both stay, with exogenous probability $\lambda \in [0, 1]$.[8]

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[6] The analysis extends to settings in which the set of feasible efforts depends on the state, as long as more effort-levels are feasible in higher states, i.e. $x_t' \succ x_t$ implies $\mathcal{E}_{it}(x_t') \supseteq \mathcal{E}_{it}(x_t)$ for all $i$.

[7] The effort-set $\mathcal{E}_{it}$ can be viewed as the set of all mixtures over some underlying set of pure efforts, endowed with the first-order stochastic dominance ordering inherited from the partial order on that underlying set. The assumption here of observable efforts then corresponds to an assumption that effort *mixtures* are observed. If mixtures cannot be observed, this paper's results can be viewed as characterizing welfare-maximizing *pure-strategy* SPE.

[8] $\lambda$ corresponds to the "rate of death" introduced in Section 5, when a re-matching technology is added to the basic model presented here.

If so, each player $i$ receives an outside option having present value $v_i \geq 0$ and nothing thereafter.[9] Otherwise, the partnership remains active in period $t + 1$.

Throughout the game, players can make voluntary wage transfers to one another at any time. However, it is without loss to restrict attention to SPE in which only "retention bonuses" are paid each period, after and only if both players decide to stay (Lemma 1). For simplicity, then, I will proceed as if this is the only time at which players can pay wages. (Wages paid after efforts are observed, in the form of "performance bonuses", do *not* provide adequate equilibrium incentives. See Example 2.)

**Solution concept.** The solution concept is subgame-perfect equilibrium (SPE), with special focus on SPE that maximize players' joint welfare among all SPE.

**Stochastic process.** The stochastic process $(X_t : t \geq 0)$ has the property that future states are more likely to be higher when the current states is higher, a serial auto-correlation property that I will refer to as "persistence". Two definitions are needed to make this precise.

**Definition 1** (Increasing subset)**.** Let $(\mathcal{Z}, \geq)$ be any partially-ordered set. $\mathcal{Y} \subset \mathcal{Z}$ is an "increasing subset of $\mathcal{Z}$" if $a_1 \in \mathcal{Y}$, $a_2 \in \mathcal{Z}$, and $a_2 \geq a_1$ implies $a_2 \in \mathcal{Y}$.

**Definition 2** (Generalized first-order stochastic dominance[10])**.** Let $A_1, A_2$ be random variables with support in partially ordered set $(\mathcal{Z}, \geq)$. $A_1$ "first-order stochastic dominates" (FOSD) $A_2$ if $\Pr(A_1 \in \mathcal{Y}) \geq \Pr(A_2 \in \mathcal{Y})$ for all increasing subsets $\mathcal{Y} \subset \mathcal{Z}$.

**Assumption 4** (Persistence)**.** $x'_t \succeq x_t$ implies $X_{t+1}|(x'_t, \mathbf{e}_t)$ FOSD $X_{t+1}|(x_t, \mathbf{e}_t)$ for all $\mathbf{e}_t$.

**Definition 3** (Cost of effort)**.** Let $c_{it}(e_t; x_t) = \sup_{\tilde{e}_{it}} (\pi_{it}(\tilde{e}_{it}, e_{jt}; x_t) - \pi_{it}(e_t; x_t))$ denote each player's "cost of effort $e_{it}$" when player $j$ exerts effort $e_{jt}$ at time $t$.

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[9] The analysis and basic results extend to settings in which players' outside options $V_t = (V_{it}, V_{jt})$ are random, as long as they are not controlled by players' efforts. (In particular, $V_t$ may be correlated with $X_t$, but only if $X_t$ follows an exogenous stochastic process as in Section 4.1.) Extending the analysis to allow for endogenous outside options, as when players may search for their next partner while still matched, is an important direction for future research.

[10] When $\mathcal{Z} = \mathbf{R}$, this condition reduces to the familiar requirement that $\Pr(A_1 \geq z) \geq \Pr(A_2 \geq z)$ for all $z \in \mathbf{R}$. There is more than one natural way to generalize FOSD to multi-dimensional settings, some more restrictive than the notion used here. See e.g. Stoyan and Daley (1983).

*Discussion of the model:* By Assumption 1, each player has a weakly dominant strategy to exert zero effort in each effort stage-game, so $c_t(e_t; x_t) = \pi_{it}(0, e_{jt}; x_t) - \pi_{it}(e_t; x_t)$. By Assumption 2, $x'_t \succ x_t$ implies

$$\pi_{it}(e_t; x'_t) \geq \pi_{it}(e_t; x_t) \text{ for all } e_t \tag{1}$$

$$c_t(e_t; x'_t) \leq c_t(e_t; x_t) \text{ for all } e_t \tag{2}$$

for all $e_t$. That is, stage-game payoffs are increasing in the state *and* players' incentive to exert less effort is decreasing in the state. (Increasing differences implies (1) when we set $e_t^H = e_t$ and $e_t^L = (0, 0)$ and implies (2) when we set $e_t^H = e_t$ and $e_t^L = (0, e_{jt})$.)

Assumption 3 requires that inside productivity and outside options be unrelated. Interactions between inside and outside payoffs can arise quite naturally, for at least two reasons. First, outside option values may be correlated with the partnership state for exogenous reasons, if there is some common factor driving both. For example, a player's outside option might be to start his own business in the same line of work as that pursued by the partnership. Second, activities that enhance a player's outside option may also enhance or detract from the future productivity of the partnership.

Assumption 4 states that the partnership is weakly more likely to transition to a "better state" tomorrow from a better state today, holding fixed the history of players' efforts. The fact that no assumptions are made on how efforts impact future states allows for great flexibility, e.g. the model can accommodate settings in which effort grows, depletes, or has a non-monotone effect on a payoff-relevant stock. On the other hand, Assumption 4 does rule out a variety of potential applications in which payoffs are stochastic but not persistent. For instance, suppose that $\mathcal{X}_t = \{\text{low}, \text{high}\}$ for all $t$ as in Bagwell and Staiger (1997). Assumption 4 fails in the case of negative serial auto-correlation.

Here are some simple examples of state processes $(X_t : t \geq 0)$ satisfying Assumption 4. In each case, $\mathcal{X}_t \subset \mathbf{R}^K$. Examples (A-C) are exogenous Markov processes, (D) is a non-Markov exogenous process, (E) is a non-trivially controlled process.

(A) $X_t$ are iid.

(B) $X_t$ reverts to the mean, e.g. $X_t = \rho X_0 + (1-\rho) X_{t-1} + \sigma \varepsilon_t$, where $\varepsilon_t$ are iid mean zero.

(C) $g(X_t)$ is a random walk where $g : \mathbf{R}^K \to \mathbf{R}^K$ is any non-decreasing function relative to the usual product order on $\mathbf{R}^K$.

(D) $X_t = (Y_0, ..., Y_t)$ is a sequence of publicly observed estimates of $K$ unobserved parameters, e.g. unknown productivity of the match à la Jovanovic (1979a).

(E) $X_t$ is a capital stock with a random growth rate, e.g. $X_{t+1} = Y_t(X_t + \sum_i e_{it})$, where $(Y_t : t \geq 0)$ is an exogenous stochastic process as in any of the previous examples.[11]

# 3 Example: Dynamic Prisoners' Dilemma

This section provides a simple example that serves to fix ideas and highlight some aspects of the more general analysis to come.

| | Work | Shirk |
|---|---|---|
| Work | $1, 1$ | $-1 - c_t, 1 + c_t$ |
| Shirk | $1 + c_t, -1 - c_t$ | $0, 0$ |

Figure 2: Stage-game payoffs at time $t$, while the partnership persists.

Stage-game payoffs each period are shown in Figure 6.[12] "Effort cost" $c_t > 0$, and $\log(C_t)$ evolves according to a known and exogenous random walk with iid motion that is atomless on support that contains $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Should the partnership dissolve, each player gets the same outside option having value $v \geq 0$. All assumptions of Section 2 are satisfied when we define the state $x_t = -c_t$.

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[11] In the joint capital accumulation game I have in mind, each player $i$ receives a dividend $d_i(X_t)$ as a function of the current capital stock, and has the opportunity to (re-)invest $e_{it} \geq 0$ to increase that stock. Overall, player $i$'s stage-game payoff at time $t$ takes the form $\pi_{it}(e_t; x_t) = d_i(x_t) - e_{it}$. The (possibly negative) rate of growth of the capital stock is $Y_t - 1$, and varies over time.

[12] Such payoffs arise naturally in a context in which players bear all of the cost of their own effort but share equally the return to that effort. Suppose that each player generates a return equal to his cost when working alone, but generates an excess return of one when working together with the other player. The payoffs of Figure 6 then arise when the cost and return of individual effort is $2(1 + c_t)$.

If $c_t = c_0$ for all $t$, then this is a standard repeated Prisoners' Dilemma.[13] One of the main points of this example is that welfare-maximizing equilibrium behavior can change dramatically once we allow $(C_t : t \geq 0)$ to follow a non-trivial stochastic process. Results to be presented later in the paper imply that a "threshold equilibrium" maximizes joint welfare among all SPE in this setting.[14] The optimal work threshold $c^{W*} = \frac{\delta}{1-\delta}$ in the standard repeated-game setting, but equals only $\frac{\delta}{2(1-\delta)}$ when $\log(C_t)$ follows any symmetric random walk with atomless transitions.

**Definition 4** (Threshold equilibrium)**.** A SPE is a "threshold equilibrium" if there exists a "work threshold" $c^W$ and "exit threshold" $c^E \geq c^W$ such that, on the equilibrium path, both players (a) work and stay if $c_t \leq c^W$, (b) shirk and stay if $c^W < c_t \leq c^E$, and (c) shirk and quit if $c_t > c^E$; off the equilibrium path, both players shirk and quit in all states.

Players' behavior in this "optimal SPE" is summarized by Figure 3. Since the outside option is fixed, the state can be viewed as moving up and down a vertical slice of this figure. When the region labeled "EXIT" is reached, both players shirk and quit. Until then, both players work and stay when in the region labeled "WORK" while both shirk and stay when in the region labeled "SHIRK". The partnership is "doomed" whenever the players' outside payoff is close enough to their continuation payoff in a productive partnership, in the sense that the players shirk and exit immediately in every state in every SPE.

**Intuitive derivation of optimal thresholds.** *Exit threshold.* Consider any threshold equilibrium with work threshold $c^W$. In such an equilibrium, the partnership can be viewed as a jointly owned asset that, prior to liquidation, generates payoff two when $c_t \leq c^W$ and

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[13] Section 5.1 provides more in-depth analysis of this example when $c_t = c_0$ for all $t$, highlighting important differences that arise should stage-game payoffs be random *across* partnerships, even as they are unchanging within each match.

[14] Since $(C_t : t \geq 0)$ is an exogenous stochastic process, joint stage-game payoff and joint continuation payoff are non-increasing in $c_t$ in a joint welfare-maximizing SPE by Theorem 3. Since joint payoff is two if both players work and zero otherwise, monotonicity of joint stage-game payoff implies that this SPE has a work threshold (that may vary with history), while monotonicity of joint continuation payoff implies that this SPE has an exit threshold (that may also vary with history). Finally, since $(\log(C_t) : t \geq 0)$ is a random walk, it is straightforward to show that the optimal work and exit thresholds do not depend on history.

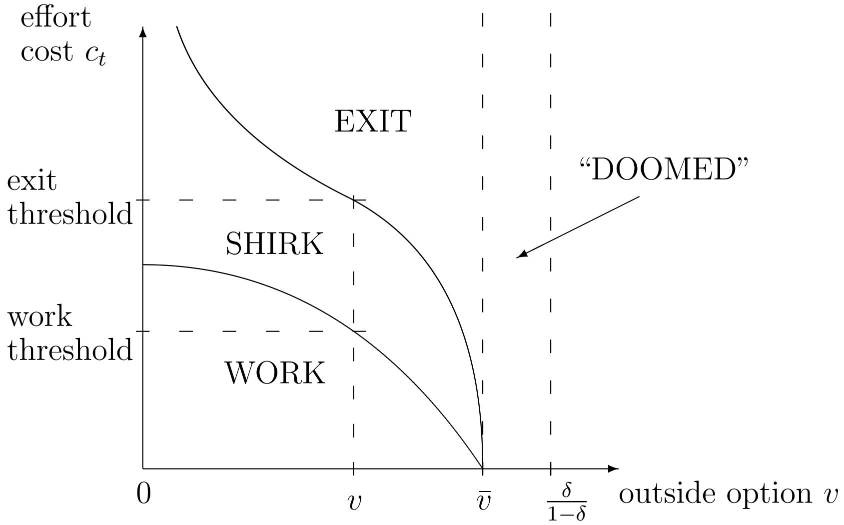

Figure 3: Summary of behavior in the optimal SPE.

zero when $c_t > c^W$, and that is worth $2v$ upon liquidation. Since $(\log(C_t) : t \geq 0)$ is a random walk, the optimal time to exercise the option to liquidate is when the state first exceeds $\alpha^* c^W$, for some $\alpha^* > 1$ that does not depend on $c^W$. Assuming that players can support work threshold $c^W$ in SPE, they can also support the optimal exit threshold given $c^W$, which is $c^E = \alpha^* c^W$. (Intuitively, equilibrium exit can be socially efficient because the players have no conflict of interest when it comes to the timing of exit, since each player gets the same outside option $v$.[15])

*Work threshold.* Threshold equilibrium payoffs are clearly increasing in the work threshold. What then is the maximal work threshold $c^W$ that can be supported in SPE, assuming an optimal exit threshold $c^E = \alpha^* c^W$? Each player can be induced to work at time $t = 0$ (or similarly at any time) only if continuing inside the partnership is more valuable than $c_0 + v$, since otherwise he would prefer to shirk and quit when the other player works. Given work threshold $c^W$ and exit threshold $c^E = \alpha^* c^W$, each player's continuation payoff $\Pi_{i0}^{cont}(c_0; c^W)$ when the current state is $c_0$ takes the form

$$\sum_{t \geq 1} \delta^t \left( \Pr\left( \max_{1 \leq s < t} C_s \leq \alpha^* c^W, C_t \leq c^W | c_0 \right) + v \Pr\left( \max_{1 \leq s < t} C_s \leq \alpha^* c^W, C_t > \alpha^* c^W | c_0 \right) \right) \quad (3)$$

$\Pi_{i0}^{cont}$ is weakly decreasing in $c_0$ by Theorem 3. Thus, to prove that both players have

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[15] In the more general analysis to come with asymmetric players, the possibility of wage transfers is essential to align the players' incentives regarding the timing of exit. Optimal wage transfers are zero here.

sufficient incentive to work when $c_0 \leq c^W$, it suffices to check that they do so when $c_0 = c^W$. Observe next that $\Pi_{i0}^{cont}(c^W; c^W)$ does not depend on $c^W$. (Intuitively, continuation payoffs depend on how frequently the state will be below $c^W$ and how quickly it will first exceed $\alpha^* c^W$. Conditional on $c_0 = c^W$, this translates as how frequently $\log(C_t)$ will be lower than $\log(c_0)$ and how quickly it will first exceed $\log(c_0)$ by more than $\log(\alpha^*)$, neither of which depends on $c^W$ by the random walk assumption.) Consequently, the maximal and hence optimal SPE work threshold is simply $c^{W*} = \Pi_{i0}^{cont}(1; 1) - v$.

**Monotonicity of thresholds.** Both the optimal work threshold $c^{W*}$ and the optimal ratio $\alpha^*$ of the exit and work thresholds are decreasing in the players' outside option $v$. Intuitively, $\alpha^*$ decreases in $v$ because staying in a temporarily unproductive relationship becomes more costly as $v$ increases, while $c^{W*}$ decreases in $v$ because a more valuable outside option combined with the prospect of quicker exit decreases the future value that can be credibly created within the partnership.

**Cooperation in fewer states.** In an unchanging version of the game, $c_t = c_0$ for all $t$, SPE exist in which players both work and stay forever as long as $c_0 \leq \frac{\delta}{1-\delta} - v$. By contrast, the optimal work threshold $c^{W*} < \frac{\delta}{1-\delta} - v$ given a changing state. The following example illustrates this basic result, which holds much more generally, whenever $(\log(C_t) : t \geq 0)$ follows a stochastic process in which all states communicate.

**Example 1.** Suppose that $v = 0$ and that $\log(C_{t+1}) - \log(C_t) \sim U[-\varepsilon, \varepsilon]$ for some $\varepsilon \geq 0$. Let $c^{W*}(\varepsilon)$ be the optimal work threshold as a function of $\varepsilon$.

**Claim 1.** *In this example, $c^{W*}(0) = \frac{\delta}{1-\delta}$ while $c^{W*}(\varepsilon) = \frac{\delta}{2(1-\delta)}$ for all $\varepsilon > 0$.*

*Proof.* As discussed earlier in the intuitive derivation of $c^{W*}$, a property of the *optimal* work threshold is that, conditional on $c_0 = c^{W*}$, each player must expect continuation payoff of *exactly* $c^{W*} + v$. Since $v = 0$, players never exit in the optimal SPE. Thus, each player's continuation payoff takes the relatively simple form

$$\sum_{t \geq 1} \delta^t \Pr\left(C_t \leq c_0 | c_0 = c^{W*}\right) = \frac{1}{2} * \frac{\delta}{1-\delta} \tag{4}$$

for all $\varepsilon > 0$, since $\Pr\left(C_t \leq c_0 | c_0 = c^{W*}\right) = \frac{1}{2}$ for all $t$ by the symmetry assumption. □

*Discussion of Claim 1.* Equilibrium cooperation is impossible whenever the players' cost of effort exceeds $c^{W*}$. Thus, conditional on $c_t = c^{W*}$, players anticipate that they will be able to cooperate at most half of the time in each future period. This shrinks by half the future value of the relationship, relative to an unchanging setting in which future cooperation can be credibly promised in every period, regardless of the "speed" $\varepsilon > 0$ at which the players' cost of effort changes over time.

Interestingly, the set of SPE payoffs is discontinuous at $\varepsilon = 0$. (For all $\varepsilon > 0$, the critical threshold for cooperation $c^{W*}(\varepsilon) = \frac{\delta}{2(1-\delta)}$, whereas the threshold $c^{W*}(0) = \frac{\delta}{1-\delta}$ in the unchanging case.) A similar discontinuity arises in Frankel, Morris, and Pauzner (2003))'s work on global games.[16] Cooperation is harder to sustain in a global game than in one with complete information because, conditional on one player's signal being exactly at the cooperation threshold, the other player's signal will be below the threshold half of the time. Similarly, cooperation is harder to sustain in the stochastic repeated game of Example 1 because, conditional on today's state being exactly at the cooperation threshold, *future* states will be below the threshold half of the time. In each case, strategic uncertainty when at the *endogenous* critical threshold determines the maximal extent of cooperation.

**Doomed partnerships.** Returning to the more general context illustrated in Figure 3, the partnership is "doomed" if both players shirk and then at least one exits at every history in every SPE (in which case the optimal work and exit thresholds $c^{W*} = c^{E*} = 0$). In an unchanging version of the game, the partnership is doomed iff $v > \frac{\delta}{1-\delta}$, since $\sum_{t\geq 1} \delta^t = \frac{\delta}{1-\delta}$ is the continuation payoff generated for each player in a permanently productive relationship. In this paper's changing environment, the partnership is doomed even when the outside option is less valuable than $\frac{\delta}{1-\delta}$. Intuitively, the reason is that partnerships always "break down before they break up". Namely, when the time is reached at which the partnership will end, both players will shirk and earn zero stage-game payoff since they have nothing to gain from exerting effort. If the players had exited earlier, they could have avoided this loss. Thus, in order to be willing to remain in the partnership, players will require that

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[16] Similar issues arise in Frankel and Pauzner (2000)'s and Burdzy, Frankel, and Pauzner (2001)'s work on dynamic coordination games.

cooperation generate an excess return over their outside option.[17]

| Period | 2 | 3 | 4 | 5 | 10 | 25 |
|---|---|---|---|---|---|---|
| % partnership ends | 25% | 16.7% | 12.8% | 9.8% | 5.0% | 2.0% |

Table 1: Probability that the partnership ends in period $t$, when $\log(C_t)$ follows a symmetric random walk, conditional on $c_0 = c^{E*}$ and on survival until time $t$.

**Survivorship bias.** Since the partnership ends once the cost of effort first exceeds an exit threshold, the partnerships that have survived several periods will, more likely than not, have received mostly positive shocks that moved the cost of effort away from the exit threshold. This positive selection effect tends to make partnerships that have lasted a long time less likely to end in the near future. For example, suppose that $\log(C_t)$ follows a symmetric random walk (as in Example 1) and that $c_0 = c^{E*}$ so that players are just barely willing to stay in the relationship. Table 1 documents the hazard rate of exit over time. For instance, conditional on survival until time $t = 4$, the partnership will end that period approximately 12.8% of the time. The survivorship bias effect is present here, as the probability of exit decreases with age.[18] (The fact that the exit hazard at time $t$ is approximately $\frac{1}{2t}$ follows from the symmetric random walk assumption; see Hughes (1995).)

# 4 Welfare-maximizing equilibrium

This section characterizes the joint welfare-maximizing subgame-perfect equilibrium (SPE) of the partnership game and highlights some of its properties. Let $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ denote the maximal joint payoff that can be achieved in *any* SPE given outside options $v = (v_i, v_j)$, as evaluated before efforts at time $t$ from payoff-relevant history $(x_t, \mathbf{e}_{t-1})$.[19] I will demonstrate

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[17] The maximal outside option $\overline{v}$ consistent with any SPE cooperation equals $\frac{\delta - X}{(1-\delta)(1+X)} < \frac{\delta}{1-\delta}$ where $X = \Pr\left(\max_{1 \le s < t} C_s \le c_0, C_t > c_0 | c_0 = c^{E*}\right) > 0$. The derivation of this formula is omitted to save space.

[18] In general, the hazard of exit need not be monotone. For instance, suppose that $\log(C_t)$ is very likely to either fall by slightly less than two or rise by slightly more than one, and that $c_0 = c^{E*}$. Conditional on both staying at time $t = 1$, the partnership is much more likely end at time $t = 3$ then at time $t = 2$.

[19] It is without loss to restrict attention to SPE in which payoffs do not depend on the history of wages.

an equilibrium that achieves this maximal SPE joint payoff at every history reached on the equilibrium path.

Each player $i$ is only willing to exert effort $e_{it}$ as part of an effort-profile $e_t$ if play (including wage transfers) after time-$t$ efforts will generate a continuation payoff of at least $v_i + c_{it}(e_t; x_t)$. In particular, costly efforts (such that $c_{\Sigma t}(e_t; x_t) > 0$) can only be sustained if joint continuation payoff inside the partnership exceeds players' joint outside option *plus* their joint cost of effort. Assuming that continuation play after time-$t$ efforts maximize players' joint continuation payoff, joint welfare is maximized given time-$t$ efforts $e_t(x_t, \mathbf{e}_{t-1})$, defined as

$$e_t(x_t, \mathbf{e}_{t-1}) = \arg\max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right]\right\}\right) \tag{5}$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \max\left\{0, \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right] - v_\Sigma\right\}. \tag{6}$$

In fact, this maximal joint payoff can be realized in SPE, with efforts $e_t(x_t, \mathbf{e}_{t-1})$ played at each history $(x_t, \mathbf{e}_{t-1})$ on the equilibrium path. Further, this equilibrium has the property that players' joint payoff is weakly increasing in the state $x_t$. (Additional comparative statics are provided in Section 4.1.)

**Theorem 1** (Joint welfare-maximizing SPE)**.** *A SPE exists that maximizes joint payoff among all SPE at every history. In this equilibrium, (i) players exert efforts* $e_t(x_t, \mathbf{e}_{t-1})$ *(defined in (5)) at every history* $(x_t, \mathbf{e}_{t-1})$*, (ii) both stay iff doing so is efficient given their joint equilibrium continuation payoff, and (iii) wages are paid only at the end of each period should both players choose to stay.*

**Theorem 2** (Welfare increasing in the state)**.** *The maximal joint welfare* $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ *that can be realized in SPE from history* $(x_t, \mathbf{e}_{t-1})$ *is weakly increasing in* $x_t$*, for all* $\mathbf{e}_{t-1}$*.*

The rest of this section provides a sketch of the proof of Theorems 1-2, as well as related discussion.

*Part I: Credibility and optimality of efforts* $e_t(x_t, \mathbf{e}_{t-1})$ *and retention bonuses.* The first part of the proof hinges on an important preliminary result.

**Lemma 1** (Joint welfare-maximizing SPE play)**.** *Suppose that SPE exist such that, at time* $t+1$ *from each history* $(x_{t+1}, \mathbf{e}_t)$*, players' joint payoff is* $\Pi^{eqm}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$*. (i) A SPE exists such that, at time* $t$ *from history* $(x_t, \mathbf{e}_{t-1})$*, players' joint payoff* $\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ *solves*

$$\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t} \left( \pi_{\Sigma t}(e_t; x_t) + \max \left\{ v_\Sigma, \delta E \left[ \Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] \right\} \right) \tag{7}$$

$$\textit{subject to } c_{\Sigma t}(e_t; x_t) \leq \max \left\{ 0, \delta E \left[ \Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma \right\} \tag{8}$$

*(ii) In this SPE, wages are paid (only) just after players decide whether to quit.*

Lemma 1 vastly simplifies the analysis since it implies that an effort-profile can be implemented in SPE iff its joint cost of effort is less than the partnership's excess return over their joint outside option. In other words, one may view partners as choosing their effort-profile optimally subject to the endogenous incentive constraint (8), i.e. they will play efforts $e_t(x_t, \mathbf{e}_{t-1})$ at every history $(x_t, \mathbf{e}_{t-1})$.

The incentive condition (8) arises naturally in repeated games with voluntary transfers. More novel here is Lemma 1's observation regarding the optimal *timing* of wages. In unchanging repeated games with exit, wages paid at any time can provide equivalent incentives to work and stay in the relationship (see Levin (2003) for a discussion). By contrast, in this paper's changing environment, optimal equilibrium wages are paid just after players' decision to stay or quit ("retention bonuses"), *not* just after their efforts ("performance bonuses"). The following example illustrates why.

**Example 2.** Consider stage-game payoffs as illustrated in Figure 4, with a state $x_t \in \{0, 1\}$ that alternates non-randomly between 0 and 1. (The state-space is unordered relative to the partial ordering defined in Section 2.) Each player has discount factor $\delta = 1/2$ and outside option $v = 6 - \varepsilon$ for some $\varepsilon > 0$. Do SPE exist in which both players stay forever, generating the highest possible joint payoff $12 + \sum_{t \geq 1} \delta^t 12 = 24$?

Suppose first that wages can be paid in the form of retention bonuses. There exists a SPE in which, each period on the equilibrium path, both players stay and the player who may earn 12 *next* period pays a retention bonus $b = 3$ after both players stay.[20] (Off the

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[20] This player is willing to pay the retention bonus since not doing so will destroy future cooperation, leading to continuation payoff $0 + v/2$, whereas paying the bonus leads to continuation payoff 9. Similarly, both

|  | Work | Shirk |
|---|---|---|
| Work | $12x_t, 12(1-x_t)$ | $0, 0$ |
| Shirk | $0, 0$ | $0, 0$ |

Figure 4: Stage-game payoffs in Example 2.

equilibrium path, both players shirk and quit.) In this equilibrium, the players' payoffs are quite different when evaluated at the start of each period. One player earns 12 and receives a retention bonus of 3, plus (as easily checked) continuation play having present value of 3, for total payoff 18. On the other hand, the other player earns 0 and must pay retention bonus 3 to enjoy continuation play having present value of 9, for total payoff 6. However, accounting for the retention bonus, each player's continuation payoff is the same at the moment of their decision to stay, allowing them to end the partnership only when it is efficient to do so given equilibrium continuation play.

Suppose next that wages must be paid as "performance wages", after stage-game payoffs are enjoyed but before the players' decision whether to quit. I will show that, for all $\varepsilon < 2/3$, no SPE exists in which the partnership lasts forever. (In fact, the partnership must end immediately.) Suppose for the sake of contradiction that a SPE exists in which both players stay forever. No player is willing to pay the other a wage greater than the difference between his own equilibrium continuation payoff and his outside option. Since each player is only willing to stay if his continuation payoff is at least $v = 6 - \varepsilon$, and joint continuation payoff is bounded above by $\sum_{t \geq 1} \delta^t 12 = 12$, each player's continuation payoff is at most $6 + \varepsilon$. Thus, no player is willing to pay a performance bonus greater than $2\varepsilon$ in any period. Consequently, the player who earns stage-game payoff 12 at time $t = 0$ expects to receive at most $2\varepsilon$ at time $t = 1$, at most $12 + 2\varepsilon$ at time $t = 2$, and so on for expected present value of at most $4 + 2\varepsilon$. This player strictly prefers to quit for outside option $v = 6 - \varepsilon$ as long as $\varepsilon < 2/3$, a contradiction. □

*Part II: Algorithmic characterization of the optimal SPE.* Next, I develop an algorithmic

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players prefer not to quit since, accounting for the retention bonus, their continuation payoffs $-3 + 18/2 = 3 + 6/2 = 6 > v$.

argument in the spirit of Abreu, Pearce, and Stacchetti (1990) (APS) to characterize the joint welfare-maximizing subgame-perfect equilibrium (SPE) of the partnership game. APS characterize the set of all SPE strategies as the limit of a decreasing sequence of sets of strategy profiles. The approach developed here differs in two ways. First, I focus on the simpler issue of characterizing just the *maximal joint payoff* that can be supported in SPE. Conceptually, at each stage of the APS algorithm,[21] identify the maximal joint payoff that can be achieved by any remaining strategy profiles. Clearly, the sequence of such upper bounds on joint payoff is decreasing and converges to the maximal SPE joint payoff. Second, and more important, the additional structure here allows me to establish new results about the joint welfare-maximizing SPE. Conceptually, by keeping track of the strategies that achieve the upper bound on joint payoffs at every step of the APS algorithm, and by showing that these strategies always satisfy certain properties, I can prove by induction that the joint welfare-maximizing SPE strategies also possess those properties. This allows me to prove, for example, that maximal SPE joint payoff is weakly increasing in the state (Theorem 2) as well as, later, additional welfare and turnover comparative statics when the state follows an exogenous stochastic process (Theorem 3).

Lemma 1 maps the maximal joint payoff that can be supported at time $t+1$ to the maximal joint payoff that can be supported at time $t$. Thus, one can construct a sequence of upper bounds for every history, $\left\{\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) : k = 1, 2, ...\right\}$, such that $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in $k$ and converges to the maximal SPE joint payoff at history $(x_t, \mathbf{e}_{t-1})$. Also importantly for this paper's purposes, these upper bounds exhibit a monotonicity in the state, i.e. $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in $x_t$ for all $k$ as well as in the limit.

$$\begin{tikzcd}
& & \mathcal{F}_t^{k+1}(x_t, \mathbf{e}_{t-1}; v) \arrow[dl, "\boxed{\mathbf{C}}"'] & & & & {} \arrow[dl] \\
{} & \overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \arrow[l] & & \overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_t; v) \arrow[ll, "\boxed{\mathbf{C}}"'] \arrow[ul, "\boxed{\mathbf{B}}"'] & & \overline{\Pi}^{k}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v) \arrow[ll, "\boxed{\mathbf{A}}"'] & {} \arrow[l]
\end{tikzcd}$$

Figure 5: Key steps of the algorithmic argument.

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[21] The approach developed in APS can be extended in a natural way to dynamic repeated games. See e.g. Chapter 5.7.1 of Mailath and Samuelson (2006).

*Key steps of the algorithmic argument:* Suppose that there exists upper bounds $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ on SPE joint payoff from all histories at all times and that these upper bounds are weakly increasing in the current state $x_t$. The essence of the proof is to use these upper bounds to derive *weakly lower* upper bounds $\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ at all histories that remain weakly increasing in $x_t$. Here in the text, I provide the inductive step to construct this sequence of upper bounds on joint payoff and show that monotonicity in $x_t$ is preserved along this sequence. In the Appendix, I prove that this sequence of bounds is weakly decreasing in $k$ and that it converges to joint payoff that can be realized in SPE.

*Step A.* Given bounds $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ on joint continuation payoff at time $t+1$, joint continuation payoff after time-$t$ efforts is bounded by

$$\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v) = \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}. \tag{9}$$

Observe that

$$E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] = \int_0^\infty \Pr\left(\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z|x_t, \mathbf{e}_t\right) \mathrm{d}z. \tag{10}$$

By presumption, $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ is weakly increasing in $x_{t+1}$. Thus, the set $\{x_{t+1} : \overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all $z$. By Assumption 4, then, each of the probability terms inside the integral in (10) is weakly increasing in $x_t$. Thus, $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in $x_t$.

*Step B.* Let $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ be the set of time-$t$ efforts that can be supported in SPE given joint continuation payoffs $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ after effort, i.e. those satisfying the IC-constraint (8) given these expected joint continuation payoffs after time-$t$ effort. Since (8) slackens with higher continuation payoffs, the fact that $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in $x_t$ implies that $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in $x_t$, relative to the set inclusion order.

*Step C.* By Lemma 1, we may define new upper bounds on time-$t$ SPE joint payoff,

$$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t \in \mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)\right). \tag{11}$$

$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in $x_t$ since both $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ and $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ are weakly increasing in $x_t$, while $\pi_{\Sigma t}(e_t; x_t)$ is weakly increasing in $x_t$ by Assumption 2.

The remainder of the proof is in the Appendix. □

## 4.1 Additional comparative statics

The model imposes essentially no restriction on how efforts can control the stochastic process driving inside payoffs. Consequently, there is little that one can say in general about how efforts in the welfare-maximizing SPE vary with the state, nor on how the history of efforts impacts equilibrium variables such as players' payoffs, efforts, and exit.[22] Indeed, although Theorem 2 established that players' joint payoff in the joint welfare-maximizing SPE is weakly increasing in the state $x_t$, neither realized joint stage-game payoff nor joint continuation payoff need be weakly increasing in $x_t$. Consequently, partners may exert lower efforts and even be more likely to exit in higher states. However, additional comparative statics are available in a notable special case, when players' efforts have *no impact* on the future distribution of payoffs.

**Definition 5** (Exogenous stochastic process)**.** $(X_t : t \geq 0)$ is an *exogenous stochastic process* if the distribution of $X_t$ depends only on $(t, x_{t-1})$.[23]

Given exogeneity, players' effort-decisions at time $t$ have no impact on the set of SPE continuation payoffs. Thus, in any joint welfare-maximizing SPE, players will choose whatever efforts maximize joint stage-game payoff, among those satisfying the relevant incentive-compatibility constraint.

**Theorem 3** (Comparative statics with an exogenous state)**.** *Suppose that* $(X_t : t \geq 0)$ *is an exogenous stochastic process. In the joint welfare-maximizing SPE, at every history reached on the equilibrium path: (i) players' joint stage-game payoff and joint continuation payoff are weakly increasing in* $x_t$*; and (ii) partnership stopping time conditional on* $x_t$ *is weakly increasing in* $x_t$*, in the sense of first-order stochastic dominance.*

Suppose that $(X_t : t \geq 0)$ is an exogenous stochastic process and that $x_t \in \mathbb{R}$ for all $t$. An immediate and potentially testable corollary of Theorem 3 is that the distribution of

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[22] In various special cases of the model, not described here because of space restrictions, one can establish additional comparative statics vis-a-vis equilibrium effort and its impact on other equilibrium variables.

[23] The current state may depend on the full history of past states. For example, if $x_t = (x_{t-1}, y_t)$ for all $t > 0$, then the distribution of $X_t$ can depend on all of the "new information" $(x_0, y_1, ..., y_{t-1})$ learned during the course of the partnership.

partnership stopping time, conditional on reaching time $t$, is weakly increasing in the players' realized time-$t$ joint stage-game payoff.

# 5 Re-matching in a partnership economy

This section embeds each partnership within a "partnership economy" in which players have the option to re-match anonymously should their current partnership end.

**Partnership economy.** There is a unit mass of atomless players, half "male" and half "female", with an equal flow of $(1-\delta)$ births and deaths each period. (By introducing gender, I allow for the possibility that certain players may be matched and re-matched into specific roles, e.g. buyer and supplier, worker and firm, entrepreneur and investor.) Each player dies with exogenous probability $(1-\delta)$ each period, where death is iid across periods, and seeks to maximize his total undiscounted expected payoff prior to death.

*Re-matching.* Any player who is newly-born at time $t+1$ or whose partnership ended at time $t$ (whether due to the death of a partner or due to endogenous exit) chooses whether to "retire" for guaranteed zero payoff or to "re-match" at matching cost $m \geq 0$ with a new partner. Further, each such re-matching is a "fresh start" in two senses. First, players know nothing about their current partner's history before their partnership, including his age, number of past partnerships, and so on. Second, partnerships are stochastically independent, in the following sense.

**Assumption 5** (Fresh start)**.** Let $\{X_t^{ij_1t_1} : t \geq t_1\}$ denote the stochastic state of a potential partnership between players $i, j_1$ at time $t$, should they have begun such a partnership at time $t_1$. For all $i, j_1, j_2, t_2 \leq t_1$, $\{X_{t+t_2-t_1}^{ij_2t_2} : t \geq t_1\}$ is iid as $\{X_t^{ij_1t_1} : t \geq t_1\}$.

Assumption 5 imposes at least two substantive economic restrictions. First, shocks to a partnership are totally idiosyncratic to the players in that partnership.[24] This rules out the possibility of economy-wide shocks (correlated across partnerships active at the same time), which would of course be interesting to study in the context of enriching existing models

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[24] For all $(i_1, j_1)$ and $(i_2, j_2)$, Assumption 5 requires that $X_t^{(i_1,j_1)}, X_t^{(i_1,j_2)}, X_t^{(i_2,j_2)}$ all be independent.

of the business cycle. Indeed, extending the present analysis to allow for correlated shocks appears to be an important and promising direction for future research. Second, shocks to a partnership have no bearing on future partnerships in which those players might participate. Thus, the "state" here does not capture any payoff-relevant characteristics of the players themselves (such as intelligence, beauty, or skills).

**Assumption 6** (Public randomization)**.** At time $t = 0$, players have access to a public randomization device, i.e. $X_0 = (Z_0, Y_0)$ where $Z_0 \sim U[0,1]$ and $Z_0$ is independent of $(Y_0, X_t : t \geq 1)$.

Access to a public randomization device is not essential. In fact, under relatively mild conditions on the stochastic process, I will show that players ignore the public randomization in all social welfare-maximizing SPE. (See the corollary after Claim 2.) However, such a device is useful to draw comparisons and contrast with the existing literature on non-stochastic repeated games with re-matching.

Each individual partnership fits in the model of Section 2, given discount factor $\delta$ and exogenous probability of separation $\lambda = 1 - \delta$. (Each player will act as if maximizing discounted payoffs relative to discount factor $\delta$, since he dies with probability $(1 - \delta)$ each period. Each player is exogenously separated iff his current partner dies, which happens with probability $1 - \delta$ conditional on his own survival.) Of course, players' outside options $v = (v_i, v_j)$ are endogenous in the partnership economy, as they depend on how players expect future partnerships to proceed.

Since past partnership histories are unobserved, it is without loss to restrict attention to equilibria of the overall partnership economy in which all pairs of players play the same SPE of the partnership game.

**Definition 6** (Partnership-economy equilibrium)**.** A *"partnership-economy equilibrium"* is a SPE of the partnership game satisfying two properties:

$$E\left[\Pi_{i0}^{eqm}(X_0; v)\right] \geq m \text{ for } i = 1, 2 \tag{12}$$

$$v_i = \delta\left(E\left[\Pi_{i0}^{eqm}(X_0; v)\right] - m\right) \text{ for } i = 1, 2 \tag{13}$$

Should (12) fail, no matches would ever occur since all players of at least one gender strictly prefer to retire at birth. Any equilibrium of the partnership game that is actually played in the overall economy must guarantee each player an expected payoff that covers the cost of matching. For a player who is currently matched at time $t$, quitting and re-matching yields a continuation payoff having present value $\delta \left(E\left[\Pi_{i0}^{eqm}(X_0; v)\right] - m\right)$. (13) is therefore just the general equilibrium condition that each player's outside option is generated endogenously by his/her option to exit and re-match.

A social welfare-maximizing partnership-economy equilibrium is one that maximizes players' joint outside option, among all partnership-economy equilibria. Recall that $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$ denotes the maximal SPE joint payoff at history $(x_t, \mathbf{e}_{t-1})$ given outside options $v = (v_i, v_j)$. Lemma 2 establishes that this maximal joint payoff depends on players' outside options only through their *sum.*

**Lemma 2.** *$v'_\Sigma = v_\Sigma$ implies $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v') = \overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.*

*Proof.* The proof of Lemma 2 is immediate from the algorithmic construction in the proof of Theorems 1-2. Lemma 2 can also be viewed as a corollary of Lemma 1, once one observes that players' outside options do not appear in the objective (7) or in the constraint (8) except through the sum $v_\Sigma$. Intuitively, asymmetries in players' outside options have no impact on what can be achieved in equilibrium, since any such asymmetries can be counter-balanced by appropriate retention bonuses. □

**Definition 7** (Maximal SPE joint payoff)**.** Let $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v_\Sigma)$ denote the maximal joint payoff in any SPE from history $(x_t, \mathbf{e}_{t-1})$, as a function of players' *joint* outside option $v_\Sigma$.

A necessary condition of partnership-economy equilibrium is that the players' joint outside option $v_\Sigma$ can be "self-generated" in *some* SPE given $v_\Sigma$:

$$v_\Sigma \leq \delta \left(E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)\right] - 2m\right). \tag{14}$$

Thus, players' joint outside option cannot exceed $\sup\{v_\Sigma : v_\Sigma \leq \delta \left(E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma^*)\right] - 2m\right)$. Theorem 4 establishes that this maximal joint outside option can in fact be supported in equilibrium, and shows one way in which to do so.

**Theorem 4** (Maximal social welfare)**.** *In social welfare-maximizing partnership-economy equilibria, players' endogenous joint outside option is*

$$v_\Sigma^* = \sup\left\{v_\Sigma : v_\Sigma \leq \delta \left(E\left[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)\right] - 2m\right)\right\}. \tag{15}$$

*Further, play in one such equilibrium proceeds as follows: if the public randomization* $z_0 \leq \frac{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^*_\Sigma)\right] - 2m - v^*_\Sigma/\delta}{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^*_\Sigma)\right] - v^*_\Sigma}$*, then both players exert zero effort and quit immediately; otherwise, continuation play maximizes SPE joint welfare as in Theorem 1.*

*Discussion of Theorem 4:* Suppose for the moment that the maximal expected joint payoff that can be supported in SPE, $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right]$, is continuous in the players' joint outside option $v_\Sigma$. In this case,

$$v^*_\Sigma = \delta\left(E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^*_\Sigma)\right] - 2m\right). \tag{16}$$

Theorem 4 then implies that equilibrium social welfare is maximized only when players maximize equilibrium joint welfare within each partnership.

This is one of the key findings of the paper. While natural and intuitive, this connection between social welfare and joint welfare is not obvious and, in fact, flies in the face of received wisdom about welfare-maximizing play in repeated games with re-matching. In non-stochastic repeated games with re-matching, maximizing social welfare always requires that partners fail to maximize joint welfare. (See Section 5.2 of Mailath and Samuelson (2006) for a particularly clear exposition.) Theorem 4 shows that this important result hinges crucially on what may have seemed like an incidental property of these games: viewed as a function of the players' joint outside option, the maximal SPE joint payoff is discontinuous at $v^*_\Sigma$.

A key insight here is that the tension between economy-wide and individual partnership performance disappears once partnerships are *not* be created equal, namely, when "first impressions" are payoff-relevant. (For an example, see Section 5.1 and especially Figures 7-8.) Indeed, even quite modest initial stochastic variation across partnerships eliminates the need to burn money or otherwise perform sub-optimally.[25] For instance, suppose that the initial state is augmented with a one-dimensional "partnership type" $s_0$, as follows.

**Definition 8** (Effort-incentivizing type)**.** The initial state $x_0$ includes an "independent effort-incentivizing partnership type" $s_0$ if (i) $x_t = (s_0, y_t)$ for all $t \geq 0$, (ii) $S_0 \in \mathbf{R}$ is atomless and independent of $(Y_t : t \geq 0)$, and (iii) $\pi_{it}(e'_t; s_0, y_t) - \pi_{it}(e_t; s_0, y_t)$ is strictly increasing in $s_0$ for all $i, t, y_t, e'_t \succ e_t$.

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[25] Theorem 4 exhibits such sub-optimal play when $v^*_\Sigma < \delta\left(E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^*_\Sigma)\right] - 2m\right)$, since players sometimes immediately give up on a relationship that could have worked.

The partnership type $s_0$ captures an aspect of "initial fit" between partners in that, all else equal, higher partnership types lead to higher inside payoffs and lower effort costs. (Both $\pi_{it}(e_t; s_0, y_t) - \pi_{it}(0, 0; s_0, y_t) = \pi_{it}(e_t; s_0, y_t)$ and $-c_{it}(e_t; s_0, y_t) = \pi_{it}(e_t; s_0, y_t) - \pi_{it}(0, e_{jt}; s_0, y_t)$ are strictly increasing in $s_0$, for all $t, x_t, e_t$.) Since partnership type can have an arbitrarily small impact on payoffs, one may view Claim 2 as establishing the continuity condition of Theorem 4 in a perturbation of the model in which "initial fit" matters. The additional assumption of Claim 2, that there are finitely many efforts, is not essential but simplifies and shortens the proof.

**Claim 2.** *Suppose that the initial state includes an independent effort-incentivizing partnership type, and that* $\mathcal{E}_t$ *is finite for all* $t$*. Then* $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is continuous in* $v_\Sigma$*.*

The following is an immediate corollary of Theorem 4 and Claim 2.

**Corollary.** *Under the conditions of Claim 2, all social welfare-maximizing partnership-economy equilibria maximize the players' expected joint welfare given the players' endogenous outside options.*

The presence of an effort-incentivizing type eliminates the need to burn money at the start of relationships. Intuitively, the reason is that observing a payoff-relevant type at time $t = 0$ breaks players' *indifference* over potential partners. When players do not care about the identity of their partner, each player will naturally be concerned that his current partner will cheat him and then re-match with an equally attractive replacement. Burning money at the start of every partnership allows players to assuage this concern. Once players strictly prefer some partners over others, however, the re-matching market no longer provides "easy pickings" for a cheater. In particular, players will reject any partner who is not a sufficiently good fit, and fear of future rejection provides a deterrent against misbehavior in any sufficiently well-matched partnership. Of course, the threshold for a "sufficiently good fit" is endogenous. It is determined so that players at this threshold are indifferent between (i) staying and playing the joint welfare-maximizing SPE of Theorem 1 or (ii) quitting to re-match. (For a worked-out example, see Section 5.1.)

When initial fit matters in the sense of Claim 2, Theorem 1 implies that all social welfare-maximizing partnership-economy equilibria specify (essentially[26]) the same joint welfare-maximizing efforts every period. More broadly, whenever $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is continuous in $v_\Sigma$, joint welfare-maximizing equilibrium play within each partnership is *necessary* to maximize equilibrium social welfare. Claim 3 establishes that joint welfare-maximizing play is also *sufficient* to maximize equilibrium social welfare in this case.

**Claim 3.** *Suppose that* $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is continuous in* $v_\Sigma$*. Then* $v^*_\Sigma$ *is the* unique *solution to*

$$v_\Sigma = \delta\left(E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right] - 2m\right). \tag{17}$$

*Discussion of Claim 3:* Say that outside options $v = (v_i, v_j)$ are "self-generated" by joint welfare-maximizing play if $v_i = \delta\left(E\left[\overline{\Pi}^{eqm}_{i0}(X_0; v)\right] - m\right)$ for $i = 1, 2$. Claim 3 implies that only one *joint* outside option can be self-generated by joint welfare-maximizing play. Technically, the proof proceeds by showing that the maximal excess joint return of a new partnership over the players' joint outside option, $\delta\left(E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)\right] - m\right) - v_\Sigma$, is strictly decreasing in $v_\Sigma$. Intuitively, as players' outside options become less valuable, they have more reason to work and invest in their relationship. Thus, even if falling outside options are bad news in the sense of lowering equilibrium payoffs, this loss is mitigated by the fact that the players' partnership becomes stronger and more productive.

While social welfare-maximizing partnership-economy equilibria all specify the same efforts each period, such equilibria may differ in how the surplus is divided between the players. For example, if a wife anticipates that her *next* husband will pay her a handsome wage, then her current husband may have to pay her a wage to induce her to stay. In this way, even if there exists a social welfare-maximizing partnership-economy equilibrium in which no wages are paid, other such equilibria may exist in which either player receives the lion's share of the surplus.

**Partnership formation costs.** Let $v^*_\Sigma(m)$ denote the maximal joint outside option that can be supported in partnership-economy equilibrium, as a function of matching cost $m \geq 0$.

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[26] If (5) has multiple solutions at some history, then there will exist different social welfare-maximizing SPE in which each of these optimal IC efforts is played. Otherwise, efforts are unique.

**Claim 4.** *Suppose that* $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is continuous in* $v_\Sigma$*. Then* $v^*_\Sigma(m)$ *is strictly decreasing in* $m$*.*

*Discussion of Claim 4*: An increase in partnership formation costs has competing effects on payoffs in the social welfare-maximizing equilibrium. While players must pay more to form each partnership, such costs can act as an exit deterrent and hence encourage players to work and invest in their current partnership. However, this benefit of better partnership performance only arises if the *overall* effect of higher partnership formation costs is to lower players' outside options. Thus, this overall effect must be at least *weakly* negative. Further, as long as "initial fit matters" (which implies continuity of $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ by Claim 2), the overall effect of higher partnership formation costs must indeed be *strictly* negative. The reason is that players will respond to higher partnership formation costs by sampling strictly fewer partners, leading to a strictly worse average fit among active partnerships.

By contrast, in a non-stochastic repeated game with re-matching in which all partnerships are equally attractive, all small enough partnership formation costs have no effect on maximal equilibrium welfare. The reason is two-fold. First, players already burn money in the social welfare-maximizing equilibrium of these games. As long as partnership formation costs are less than the optimal amount of burned money, such costs merely substitute for money burning. Second, since match quality is irrelevant, players do not need to search extensively to find a good fit.

**Steady-state distribution over partnership histories.** One may view a partnership in the economy as a Markov chain over *histories* $h_t = (x_t; \mathbf{e}_{t-1})$, where any partnership that ends at time $t$ is understood to transition to a brand new partnership.

Suppose that a partnership is currently in history $h_t$. Let $e_t(h_t)$ be the effort-profile played in the optimal SPE at this history, as characterized in the proof of Theorem 2. Similarly, let $p^{exit}_t(h_t)$ be the probability of endogenous exit, due to at least one player choosing to leave, and let $X_{t+1}(h_t) \sim X_{t+1}|(h_t, e_t(h_t))$ denote next-period's state should the partnership persist to that time. Transition probabilities among histories may be fully described as follows:

- With probability $1 - \delta^2$, the partnership will end due to death, after which a new partnership will be created having random initial history $H_0 = X_0$.

- With probability $\delta^2 p_t^{exit}(x_t, \mathbf{e}_{t-1})$, the partnership will end due to some partner's endogenous departure, after which a new partnership will again be created.

- With probability $(1-\delta^2)(1-p_t^{exit}(x_t, \mathbf{e}_{t-1}))$, the partnership will continue to time $t+1$, with an augmented random history $H_{t+1} = (h_t; e_t(h_t); X_{t+1}(h_t))$.

Note that, through the process of death and re-birth, all histories that are reached on the equilibrium path communicate and are positively recurrent. Thus, this Markov chain is ergodic and there exists a unique steady-state distribution over histories. (The proof of Claim 5 is omitted to save space. See Sections 4.3 and 4.6.2 of Ross (1996), especially Theorem 4.3.3.)

**Claim 5** (Steady-state distribution)**.** *For every partnership-economy equilibrium, there exists a unique steady-state distribution over partnership histories.*

In the remainder of this section, I discuss some qualitative features of a "typical" player's life experience, assuming play of a welfare-maximizing partnership-economy equilibria.

**Dating.** At time $t = 0$, players will immediately exit any relationship in which the realized initial state is in a decreasing subset of $\mathcal{X}_0$. Consequently, any player who is seeking a new partner will typically experience several partnerships that each last exactly one period – and in which both players exert zero effort because they anticipate no future interaction – before finding a partner who they do not immediately leave.

**Honeymoon.** In any partnership that continues to a second period, players obviously expect higher continuation payoffs than during their unsuccessful dating phase. In fact, such "newly-joined" partners will also enjoy higher stage-game payoffs than when they were unsuccessfully dating, for two reasons. First, the initial state in a "successful date" will be higher than in an unsuccessful one, allowing players to generate higher stage-game payoffs from any time-0 efforts (Assumption 2). Second, since the players view their future relationship as generating higher continuation payoffs than their outside options, they can also support non-trivial effort at time $t = 0$.

There is no guarantee that a partnership in its "honeymoon" will be very profitable or very stable. For instance, it could be that the initial state lies very close to the threshold below which the partnership would not have formed, and that there is a high likelihood of break-up in the near future. On the other hand, depending of course on the distribution of the initial state, a large fraction of new partnerships may have initial states far enough above this threshold so that exit is very unlikely for several periods. If so, one would observe a "dating and honeymoon effect" in which partnerships are very likely to end in their first period, very unlikely to end in their second period, and then somewhat more likely to end over the next several periods. Fichman and Levinthal (1991) have previously articulated the honeymoon effect in a related non-strategic context.[27]

**Hard times.** The state of a partnership may rise and fall many times, in ways that affect the extent of cooperation that can be supported in the welfare-maximizing equilibrium. This volatility of players' willingness to cooperate creates payoff volatility that in turn creates an endogenous option value to remaining in the relationship. Consequently, players tend to remain in relationships even when stage-game payoffs are low, in hopes that their partner's behavior will improve.

This feature of equilibrium behavior appears consistent with an interesting fact about marital separation in the United States, if one can interpret reports of "happiness" as reflecting stage-game payoffs. The National Survey of Families and Households (NSFH) of 1987-1988 asked about two thousand individuals who had experienced marital separation relatively recently to evaluate their experience.[28] When comparing their overall happiness "now, compared to the year before you separated", 57.8% described their current happiness as "much better" while only 2.9% described it as "much worse." One possible explanation of this survey result is that spouses' expectations regarding the future performance of a marriage (relative to their outside options) changes over time, so that there is an option

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[27] Fichman and Levinthal (1991) consider an organization's decision to form and disband, when profits follow an exogenous random walk. The analysis here differs in several ways, the most important being that profits are endogenous.

[28] All respondents had experienced a separation since January 1, 1977. See www.ssc.wisc.edu/nsfh, accessed June 4, 2009.

value of staying married.[29] This option value could arise from exogenous variation in the fundamentals of the marriage ("perhaps my wife will get a raise") and/or from endogenous variation in spousal behavior ("perhaps my husband will stop cheating on me").

**Good times and golden years.** Players stay in the partnership during hard times in the hope that they will enjoy positive shocks that will enable them to enjoy higher profits and greater stability in the future. Indeed, depending on the details of the stochastic process $(X_t : t \geq 0)$, there may be an increasing subset of the state-space from which the partnership is certain never to end, save by exogenous death. Such "golden years" can arise for two sorts of reasons. First, there may be an absorbing portion of the state-space, that is everywhere high enough to support continuation of the partnership. Second, equilibrium efforts in high enough states may be sufficiently high and feedback from profitable efforts may be positive enough to overwhelm any exogenous shocks that might cause the relationship to deteriorate.

## 5.1 Example: Repeated Prisoners' Dilemma

In this section, I re-consider classic results on social welfare-maximizing equilibria in the repeated Prisoners' Dilemma with re-matching, in light of this paper's findings. Recall that $m \geq 0$ is the cost of matching or re-matching, while $Z_0$ is a public randomization device.

Payoffs are as in the example of Section 3, except that stage-game payoffs do not change over time within any given partnership. However, the "cost of effort" $c > 0$ may be random *ex ante* and iid across partnerships. All of this paper's assumptions on payoffs and the stochastic process are satisfied trivially, when the "state" is just the cost of effort.

| | Work | Shirk |
|---|---|---|
| Work | $1, 1$ | $-1-c, 1+c$ |
| Shirk | $1+c, -1-c$ | $0, 0$ |

Figure 6: Stage-game payoffs at time $t$, while the partnership persists.

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[29] Other plausible explanations have nothing to do with uncertainty about how the marriage will perform, such as selection bias (if happier individuals are more likely to be surveyed) or choice-supportive bias (if respondents tend to remember the past in ways that help justify their decisions).

**Non-random cost of effort.** To begin, consider the classic case in which stage-game payoffs are non-random. I will solve here for the maximal social welfare that can be supported in partnership-economy equilibrium (called $v^*$), viewing the non-random cost of effort $c > 0$ and matching cost $m \geq 0$ as parameters. First, a useful preliminary result, the proof of which is straightforward and omitted to save space.

**Claim 6.** *Suppose that each player has outside option* $v \geq 0$.[30] *Once matched in a partnership with cost of effort c, the joint welfare-maximizing SPE characterized by Theorem 1 proceeds as follows: (i) if* $c \leq \frac{\delta}{1-\delta} - v$*, each player works and stays forever for payoff* $\frac{1}{1-\delta}$*; (ii) otherwise, each player shirks and quits immediately for payoff* $v$*.*

*Case #1: No partnership formation (*$c > \frac{\delta}{1-\delta}$ *or* $m > \frac{1}{1-\delta}$*).* If $c > \frac{\delta}{1-\delta}$, Claim 6 implies that players shirk and quit immediately in every partnership, earning zero payoff once matched. In this case, $v^* = 0$ and welfare is maximized when players choose never to match. Suppose next that $c \leq \frac{\delta}{1-\delta}$ but that $m > \frac{1}{1-\delta}$. In this case, the maximal possible joint payoff $2 + \sum_{t \geq 1} 2\delta^t = \frac{2}{1-\delta}$ is not enough to support both players' matching costs. So, again, social welfare is maximized when players are unmatched.

*Case #2: One permanent partnership (*$c \leq \frac{\delta}{1-\delta}$ *and* $m \in \left[\frac{c}{\delta}, \frac{1}{1-\delta}\right]$*).* Suppose next that $c \leq \frac{\delta}{1-\delta}$ and $m \in \left[\frac{c}{\delta}, \frac{1}{1-\delta}\right]$. Assuming that players will work and stay forever in their *next* partnership, each player's outside option in his current partnership $v = \delta\left(\frac{1}{1-\delta} - m\right) \in \left[0, \frac{\delta}{1-\delta} - c\right]$. Thus, by Claim 6, players can indeed support working and staying forever in their first partnership in a partnership-economy equilibrium. Obviously, this equilibrium maximizes social welfare.

*Case #3: Search as burning money (*$c \leq \frac{\delta}{1-\delta}$ *and* $m \in \left[0, \frac{c}{\delta}\right]$*).* When matching cost $m < \frac{c}{\delta}$, players must "burn money" between partnerships to provide sufficient incentives to work in any relationship. (If players could transition costlessly between productive partnerships, each player's outside option $v = \delta\left(\frac{1}{1-\delta} - m\right) > \frac{\delta}{1-\delta} - c$, in which case no player would be willing to work.) In one social welfare-maximizing equilibrium, this failure to achieve

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[30] In the context of this section's example, it is without loss to restrict attention to equilibria with symmetric outside options.

the full potential gains from one's partnerships takes the form of *excessive search* in which players pay the matching cost more than once and fail to exert any effort during each search interlude. (If $m = 0$, search is still costly as players endure an interlude with zero stage-game payoffs.)

**Claim 7.** *Suppose that* $c \leq \frac{\delta}{1-\delta}$ *and* $m \in \left[0, \frac{c}{\delta}\right]$. *Play proceeds as follows in a social welfare-maximizing partnership-economy equilibrium: (i) if the randomization* $z_0 < \frac{c/\delta - m}{1+c}$, *then players shirk, quit immediately, and re-match; (ii) otherwise, players work and stay forever. The players' outside option* $v^* = \frac{\delta}{1-\delta} - c$ *in this equilibrium.*

*Proof.* To support outside option $v$ in some partnership-economy equilibrium, each player's payoff in some SPE must be at least $\frac{v}{\delta}$. In particular, no outside option $v > \max\{0, \frac{\delta}{1-\delta} - c\}$ is supportable: given any such outside option, it is straightforward to check that both players shirk and the partnership ends immediately in all SPE, for payoff only $v < \frac{v}{\delta}$. Thus, $\frac{\delta}{1-\delta} - c$ is an upper bound on the outside option that can be supported in equilibrium.

Suppose now that players adopt the strategies specified in Claim 7 (augmented by shirking and quitting at any history off the specified path of play). At birth, each player earns expected payoff $\hat{\Pi}$, where

$$\hat{\Pi} = \frac{1}{1-\delta}\left(1 - \frac{c/\delta - m}{1+c}\right) + \delta\hat{\Pi}\frac{c/\delta - m}{1+c} - m \tag{18}$$

has solution $\hat{\Pi} = \frac{1}{1-\delta} - \frac{c}{\delta}$, generating the maximal outside option $v^* = \frac{\delta}{1-\delta} - c$ whenever a player leaves his current match.[31]

To complete the proof, it suffices to check that these specified strategies constitute a SPE of the partnership game given outside option $v^*$. First, each player prefers to stay in the partnership when he expects permanent cooperation, since his inside continuation payoff $\frac{\delta}{1-\delta} > v^*$. Similarly, each player is indifferent between working (and staying) and shirking (and quitting) since $c + \frac{\delta}{1-\delta} = v^*$, and so is willing to work. Finally, each player is obviously willing to shirk and quit when the randomization $z_0 < \frac{\delta}{1-\delta} - c$, since players in such a partnership expect their partner to shirk forever should the relationship persist. $\square$

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[31] FOR REFEREES ONLY: to save you time, note that (18) can be re-written as $\hat{\Pi} = \frac{1}{1-\delta} - \frac{c/\delta - m}{1+c}\left(\frac{1}{1-\delta} - \delta\hat{\Pi}\right) - m$. Now, letting $\hat{\Pi} = \frac{1}{1-\delta} - \frac{c}{\delta}$, observe that $\frac{c/\delta - m}{1+c}\left(\frac{1}{1-\delta} - \delta\hat{\Pi}\right) = \frac{c}{\delta} - m$.

**Random cost of effort.** Now suppose that the cost of effort, while unchanging in any given partnership, is iid across partnerships with cdf $F(.)$.

*Case #1: No partnership formation* ($m > \frac{1}{1-\delta}F\left(\frac{\delta}{1-\delta}\right)$). Players strictly prefer not to match at all when the matching cost is high enough, with a critical threshold cost $m^* = \frac{1}{1-\delta}F\left(\frac{\delta}{1-\delta}\right)$.[32] To see why, suppose for the sake of contradiction that $m > m^*$ and that players are willing to match in the social welfare-maximizing partnership-economy equilibrium. Let $v \geq 0$ be the endogenous outside option in this equilibrium.

Matched players can support cooperation (for the maximal possible continuation payoff $\frac{1}{1-\delta}$) only when $c \leq \frac{\delta}{1-\delta} - v$; otherwise, each player strictly prefers to shirk. Thus, each player's expected payoff at birth is *at most*

$$\frac{1}{1-\delta}F\left(\frac{\delta}{1-\delta} - v\right) + v\left(1 - F\left(\frac{\delta}{1-\delta} - v\right)\right) - m < v.$$

However, since $v \leq \frac{v}{\delta}$ for all $v \geq 0$, equilibrium play cannot support the purported outside option $v$, a contradiction.

*Case #2: Joint welfare-maximizing partnerships* ($m \leq \frac{1}{1-\delta}F\left(\frac{\delta}{1-\delta}\right)$). What is the maximal outside option that can be generated in partnership-economy equilibrium, when the cost of matching is below the threshold for partnership formation?

**Claim 8.** *Suppose that the players' cost of effort is atomless and iid across partnerships, with cdf* $F(.)$, *and that the cost of matching* $m \leq \frac{1}{1-\delta}F\left(\frac{\delta}{1-\delta}\right)$. *In a social welfare-maximizing partnership-economy equilibrium, each player's outside option* $v^*(m)$ *is (uniquely) implicitly defined by*

$$v^*(m) = \frac{\frac{\delta}{1-\delta}F\left(\frac{\delta}{1-\delta} - v^*(m)\right) - \delta m}{1 - \delta\left(1 - F\left(\frac{\delta}{1-\delta} - v^*(m)\right)\right)}, \tag{19}$$

*where* $v^*(m)$ *is strictly decreasing in* $m$. *Further, in this equilibrium, players achieve their maximal SPE joint payoff given the endogenously determined outside option.*

*Proof.* Suppose that $v \geq 0$ is an outside option that can be supported in partnership-economy equilibrium. By Claim 6, an upper bound on *ex post* per-player payoff is $v$ for all $c > \frac{\delta}{1-\delta} - v$

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[32] Not coincidentally, $m^*$ is a player's expected payoff should he and his first partner be forced to match permanently, and play the joint welfare-maximizing SPE of Claim 6.

and $\frac{1}{1-\delta}$ for all $c \le \frac{\delta}{1-\delta} - v$. Thus, an upper bound on *ex ante expected* per-player payoff given this outside option is

$$\overline{\Pi}(v) = \frac{1}{1-\delta} F\left(\frac{\delta}{1-\delta} - v\right) + v\left(1 - F\left(\frac{\delta}{1-\delta} - v\right)\right) - m. \tag{20}$$

In order for outside option $v$ to be generated endogenously, it must be that $\overline{\Pi}(v) \ge \frac{v}{\delta}$. Observe that $\overline{\Pi}(v)$ is continuous in $v$ (because the cost of effort is atomless) while $\overline{\Pi}(v') - \overline{\Pi}(v) \le v' - v < \frac{v'}{\delta} - \frac{v}{\delta}$ for all $v' > v$. Thus, $\overline{\Pi}(v) = \frac{v}{\delta}$ has at most one solution. Further, since we have assumed here that $m \le \frac{1}{1-\delta} F\left(\frac{\delta}{1-\delta}\right)$, $\overline{\Pi}(0) \ge 0$ while $\overline{\Pi}\left(\frac{\delta}{1-\delta}\right) = \frac{\delta}{1-\delta} - m < \frac{1}{1-\delta}$. Thus, $\overline{\Pi}(v) = \frac{v}{\delta}$ has a unique solution $v^*(m) \in \left[0, \frac{\delta}{1-\delta}\right)$, which is an upper bound on the per-player outside option that can be supported in equilibrium. Setting $\overline{\Pi}(v) = \frac{v}{\delta}$ in (20) and re-arranging, this upper bound solves:

$$v^*(m) = \frac{\frac{\delta}{1-\delta} F\left(\frac{\delta}{1-\delta} - v^*(m)\right) - \delta m}{1 - \delta\left(1 - F\left(\frac{\delta}{1-\delta} - v^*(m)\right)\right)}. \tag{21}$$

It remains to show that $v^*(m)$ can be supported in a partnership-economy equilibrium. Yet this is immediate, when we consider SPE play within each partnership that maximizes joint payoff given this outside option. Each player earns payoff $\frac{1}{1-\delta}$ whenever $c \le \frac{\delta}{1-\delta} - v^*(m)$ and payoff $v^*(m)$ otherwise, for ex ante expected payoff $\frac{1}{1-\delta} F\left(\frac{\delta}{1-\delta} - v^*(m)\right) + v^*(m)\left(1 - F\left(\frac{\delta}{1-\delta} - v^*(m)\right)\right) - m = \frac{v^*(m)}{\delta}$ by (20). This completes the proof. □

**Simple illustration.** Consider the special case of costless matching ($m = 0$) and discount factor $\delta = \frac{1}{2}$. Suppose that the cost of effort $c$ is non-random. By Claim 7, the maximal outside option that can be endogenously supported is $v^* = 1 - c$. However, given this outside option, players are capable of supporting an outside option as large as 1. To deter players from shirking and quitting, each player therefore endures a lengthy search process, foregoing expected profit of $2c$ before starting a productive relationship. (See Figure 7.)

Suppose next that the cost of effort is random and drawn uniformly from $[0, 1]$, so that $F(c) = c$. By Claim 8, each player's outside option $v^*$ in the social welfare-maximizing equilibrium satisfies $v^* = \frac{1-v^*}{1-\frac{1}{2}(1-v^*)}$, i.e. $v^* = 2 - \sqrt{2}$. The critical threshold for the cost of effort $c^* = 1 - v^* = \sqrt{2} - 1$. Each player's ex ante expected payoff in this equilibrium is

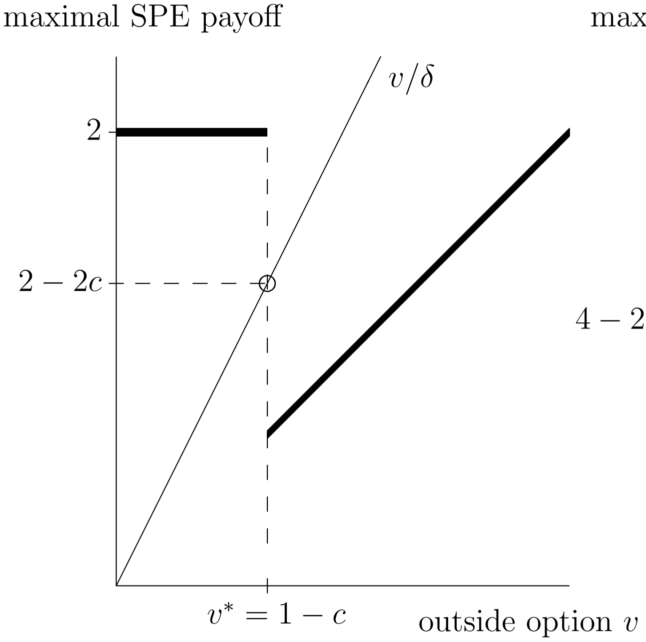

Figure 7: Non-random cost $c<1$ with $m=0$ and $\delta=1/2$.

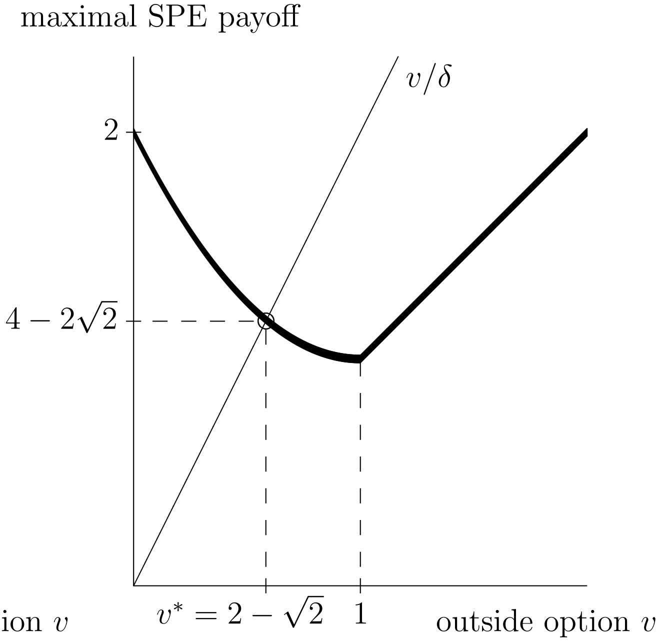

Figure 8: Random cost $c \sim U[0,1]$ with $m=0$ and $\delta=1/2$.

$4-2\sqrt{2}$. (See Figure 8, which again graphs each player's maximal expected payoff from a new match as a function of the outside option.[33])

Both players work and stay forever in any partnership in which the cost of effort $c \leq \sqrt{2}-1$, while they both shirk and quit immediately from any partnership in which the cost of effort $c > \sqrt{2}-1$. As a reality check, we can easily confirm that such play constitutes a joint welfare-maximizing SPE of the partnership game given outside option $v^* = 2-\sqrt{2}$. First, suppose that $c < \sqrt{2}-1$, in which case each player enjoys payoff $\sum_{t\geq 0}\delta^t = 2 > v^*$ with permanent cooperation. Each player prefers to stay since his inside continuation payoff $\frac{\delta}{1-\delta} = 1 > v^*$, and *strictly* prefers to work since $\frac{\delta}{1-\delta} = 1 > c+v^*$. Second, suppose that $c > \sqrt{2}-1$. In this case, it is impossible to support any cooperation in SPE. Indeed, to induce both players to work in SPE in any given period, each player must expect continuation payoff worth at least $c+v^* > 1 = \frac{\delta}{1-\delta}$, a contradiction.[34]

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[33] The upper bound on payoffs $\overline{\Pi}(v) = v^2-2v+2$ for all $v \leq 1$ and $\overline{\Pi}(v) = v$ for all $v > 1$.

[34] Since the endogenous outside option $v^*$ is worth less than the prospect of permanent future cooperation

# 6 Concluding Remarks

In their study of relational contracts in developing economies, Johnson, McMillan, and Woodruff (2002) emphasize the importance of *established* relationships in supporting the "trust" necessary to work together in an environment lacking a reliable court system. The theory of repeated games with re-matching has advanced two alternative explanations for why players may only trust those with whom they already have a working relationship. According to one view (see e.g. Kranton (1996)), social custom may require that players incur significant costs ("burn money") when establishing a relationship. Since players already in a relationship prefer to avoid burning money a second time, they will be careful to treat their current partner well. According to a second view (see e.g. Sobel (1985)), players' actions may signal information about their motives to their current partner, so that surviving partnerships are only those in which both players have proven themselves sufficiently trustworthy.

One of the key insights of this paper is that burning money is never socially optimal when (i) players have no private information and (ii) players form meaningful "first impressions" that are at least somewhat informative of future payoffs. Put differently, increasing the cost of *forming* a new relationship unambiguously lowers social welfare under these conditions, a finding that contrasts with classic results on repeated games with re-matching. Thus, this paper sheds light on the set of circumstances in which we should expect costly courtship. In addition to settings with private information, in which a suitor may feel compelled to prove his love, courtship may arise in environments in which players can only learn about the quality of their match some time *after* forming a new relationship.

Separately, a broad empirical literature from Topel and Ward (1992) on employment, Levinthal (1991) on firm survival, and Stevenson and Wolfers (2007) on marriage have established certain stylized facts about relationship dynamics. For instance, partnerships often exhibit a "honeymoon effect" and a "survivorship bias" in that very young and very old partnerships are often more profitable and less likely to end in the near future than those of

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($\frac{\delta}{1-\delta} > v^*$), $2\frac{\delta}{1-\delta}$ is an upper bound on joint continuation payoff under *any* continuation strategies. Thus, at least one player must have continuation payoff less than or equal to $\frac{\delta}{1-\delta}$.

intermediate age.[35]

A rich theoretical literature has provided a foundation for such dynamics in a context with one-sided incentives. For instance, in a labor search context (e.g. Pissarides (1994)), workers will only start a new job and/or leave their current job when presented with a sufficiently attractive new opportunity, so that new jobs will tend at first to be highly productive honeymoons. Similarly, when each firm's productivity is subject to persistent random shocks (e.g. Jovanovic (1982)), longer-lived firms will tend to be those that have received mostly positive shocks and hence be more likely to survive in the near future. This paper extends this existing literature by adding two-sided incentives and a rich stochastic structure. The resulting theory generates potentially testable predictions about the dynamics and duration of partnerships (and of search interludes between matches) in novel applications ranging from supply-chain and customer relationships to joint ventures, as well as potentially enriching the study of classic applications in labor, macroeconomics, etc.

I conclude with a brief discussion of a few interesting potential directions for future work.

*Macroeconomic volatility.* One promising direction for future work would be to consider the macroeconomic implications of this model of the economy, allowing for the more general possibility that performance is correlated across active partnerships and that re-matching is costly. For example, suppose that different partnerships that are active at the same time are subject to common shocks as well as idiosyncratic private shocks, but that the cost of re-matching does not change over time. One conjecture in this context is that equilibrium partnership dynamics will have a dampening effect on macroeconomic shocks. After a string of positive common shocks, partnerships will generally be very profitable, potentially making the cost of re-matching negligible relative to the productivity differences across partnerships. In this case, partnerships will tend to be less stable, dampening the benefits of positive

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[35] The honeymoon effect appears to be less empirically robust, e.g. while it appears to be consistent with patterns of firm performance and survival, many young marriages dissolve quickly. Stinchcombe (1965) argued that partnerships can be especially unstable when they are young if, among other reasons, players are uncertain about each other's type and quickly learn whether they are a good match. Indeed, this paper shows that Stinchcombe's insight applies even when what players quickly learn is not very informative of match quality. (For instance, in the example of Section 5.1, many partnerships end after one period even when the cost of effort is *almost* non-random.)

common shocks. On the other hand, after a string of negative shocks, the cost of re-matching may loom large enough that players' outside option may simply be to go out of business. In this case, current partners have more incentive to stay and work together, dampening the negative effect of negative common shocks.

*Changing individuals.* In this paper, each player's next partnership is stochastically identical to his current one. In other words, all shocks are to partnerships, not to the individuals in those partnerships. Of course, individuals may also change in ways that will persist in a new match. Enriching the model to allow for such personal characteristics is important and could have profound implications for the steady-state distribution of the partnership economy. For one thing, the set of players seeking to re-match will be adversely selected. This could increase active partners' desire to avoid the re-matching market, creating a still deeper adverse selection in this market.

*Endogenous learning.* The model here can capture a wide array of "learning" settings in the spirit of Jovanovic (1979a), including ones in which players make investments to increase the precision of a public signal about an unobserved payoff-relevant parameter. (Such investment could be one aspect of players' multi-dimensional effort.) When investing in a more precise signal of the underlying state, players create short-term volatility in their beliefs about the state. Such volatility can increase the value of the players' option to exit, but could also be harmful if it disrupts an otherwise productive partnership. This suggests a speculation, that players in a stable relationship may actively seek to avoid uncovering new information, while players in a rocky relationship may seek to uncover as much new information as possible.

# Appendix

## Proof of Lemma 1.

*Proof.* Let $\Pi^{eqm}_{it}(x_t, \mathbf{e}_t; v) = \delta E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]$ be shorthand for player $i$'s expected time-$t$ continuation payoff, after efforts $e_t$ from history $(x_t, \mathbf{e}_{t-1})$, should it remain active at time $t+1$. Figure 9 illustrates the key idea of Lemma 1. As long as $\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_t; v)$ exceeds the players' joint payoff after exit, $v_\Sigma$, *plus* their joint incentive to shirk from efforts $e_t$,

$c_{\Sigma t}(e_t; x_t)$, there exists a retention bonus promise given which both players have sufficient incentives to exert efforts $e_t$ and then stay. Further, this promise is credible since each player promises less than his willingness to pay to avoid cooperation breakdown.

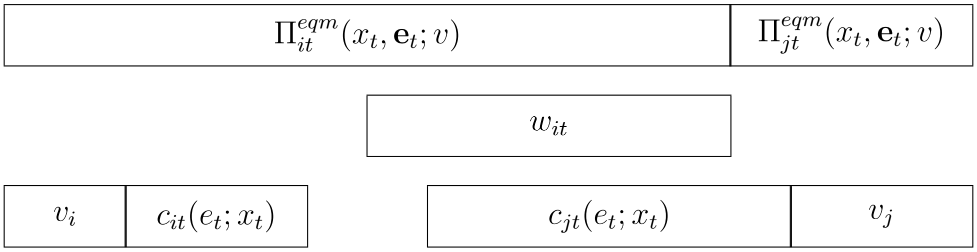

Figure 9: Efforts $e_t$ are incentive-compatible when player $i$ pays wage $w_{it}$ (and $w_{jt} = 0$).

Let $\Delta_{it}(e_t) = \Pi_{it}^{eqm}(x_t, \mathbf{e}_t; v) - v_i - c_{it}(e_t; x_t)$ denote player $i$'s "excess inside continuation payoff", the extra profit that he enjoys inside the partnership after efforts $e_t$, relative to deviating with zero effort and then quitting the relationship. $\Delta_{it}(e_t)$ is the most that player $i$ can credibly promise to pay player $j$ as a reward for not deviating from the prescribed efforts $e_t$ and then not quitting.[36]

Without loss, suppose that $\Delta_{it}(e_t) \geq \Delta_{jt}(e_t)$. If $\Delta_{it}(e_t) + \Delta_{jt}(e_t) < 0$, then at least one player must strictly prefer to deviate by exerting zero effort and then quitting, given any credible wage. Otherwise, any retention bonus $w_{it} \in [\max\{0, -\Delta_{jt}(e_t)\}, \Delta_{it}(e_t)]$ from player $i$ to player $j$ can credibly support efforts $e_t$. Thus, effort-profile $e_t$ can be supported in some SPE iff it satisfies (8). This completes the proof, since then the maximal SPE joint welfare given the specified continuation payoffs is the solution to (7). □

## Proof of Theorems 1-2.

*Proof.* Let $\overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v) \in \mathbf{R}^2$ be the players' payoff profile in a SPE that maximizes *joint* welfare among all SPE from history $(x_t, \mathbf{e}_{t-1})$, and let $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v) = \Sigma \overline{\Pi}_{it}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.

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[36] Should efforts $e_t$ be played, player $i$ becomes willing to pay up to $\Delta_{it}(e_t) + c_{it}(e_t; x_t)$ to avoid exit. Then, should both players stay to period $t+1$, player $i$ becomes willing to pay more still to avoid a transition to an optimal punishment continuation SPE in which both players exert zero effort, pay zero wages, and exit for certain at time $t+1$. Thus, as long as player $i$ has not promised to pay more than $\Delta_{it}(e_t)$, he has sufficient incentive to exert his prescribed effort, then stay, then pay the specified bonus.

*Outline of proof.* I will construct a monotonically decreasing sequence of bounds on SPE joint welfare from each history, $(\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) : k \geq 0)$, that converges pointwise to $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$, and show that this maximal joint payoff is implemented by SPE strategies as specified in Theorem 1. Further, $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in $x_t$ for each $k$, as well as in the limit $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$, establishing Theorem 2.

*Part I: Decreasing sequence of bounding payoff-profile sets.* By Assumption 1, there exists a uniform upper bound $M$ on players' joint payoff at any history. Define $\overline{\Pi}^{0}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = M$ at all histories. Clearly, $\overline{\Pi}^{0}_{\Sigma t} \geq \overline{\Pi}^{eqm}_{\Sigma t}$. Next, for all $k \geq 1$, define $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ recursively as follows (using shorthand $\mathbf{e}_t = (\mathbf{e}_{t-1}, e_t)$):

$$\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t} \left( \pi_{\Sigma t}(e_t; x_t) + \max \left\{ v_\Sigma, \delta E \left[ \overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] \right\} \right) \tag{22}$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \max \left\{ 0, \delta E \left[ \overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma \right\} \tag{23}$$

Assuming that $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ are upper bounds on joint payoff at time $t+1$, then (23) is a necessary condition for efforts $e_t$ to be supported in any SPE from history $(x_{t+1}, \mathbf{e}_t)$. *Proof:* Players at time $t$ expect joint inside continuation payoff of at most $\delta E \left[ \overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right]$ should they choose effort-profile $e_t$. If players' joint outside option $v_\Sigma$ exceeds this bound, then at least one player strictly prefers to quit and neither player can be incentivized to exert any costly effort. Otherwise, players' joint cost of effort $c_{\Sigma t}(e_t; x_t)$ must be less than or equal to the amount by which their joint inside continuation payoff exceeds their joint outside option. (Otherwise, at least one player would strictly prefer to deviate by exerting zero effort and then quitting.)

Indeed, (23) is also a sufficient condition to support time-$t$ efforts $e_t$ in SPE given continuation payoffs $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$. *Proof:* Players are willing to exert efforts $e_t$ and then receive net wage $z_{it} = w_{jt} - w_{it}$, rather than exerting zero effort, receiving zero net wage, and quitting, as long as $c_{it}(e_t; x_t) \leq E \left[ \overline{\Pi}^{k-1}_{it+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma - z_{it}$ for all $i$. In particular, efforts $e_t$ are supported by a credible promise of net wage

$$z_{it} = \frac{\left( \delta E \left[ \overline{\Pi}^{k-1}_{jt+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_j - c_{jt}(e_t; x_t) \right) - \left( \delta E \left[ \overline{\Pi}^{k-1}_{it+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_i - c_{it}(e_t; x_t) \right)}{2}$$

Given these wages (received only after both players commit to remain in the relationship), each player $i$ prefers not to quit since, given (23),

$$\delta E\left[\overline{\Pi}_{it+1}^{k-1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] + z_{it} \geq v_i + c_{it}(e_t; x_t) \geq v_i$$

(Indeed, since these "retention bonus" wages give each player the same excess continuation payoff should they stay in the partnership, players only quit their relationship when it is efficient to do so given equilibrium continuation payoffs.)

Since $\overline{\Pi}_{\Sigma t}^{0}(x_t, \mathbf{e}_{t-1}; v)$ are uniform upper bounds on joint payoff, $\overline{\Pi}_{\Sigma t}^{0}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}_{\Sigma t}^{1}(x_t, \mathbf{e}_{t-1}; v)$. By induction, $\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in $k$. (The value of the maximization (22) is non-decreasing in continuation payoffs. Thus, $\overline{\Pi}_{\Sigma t+1}^{k}(x_{t+1}, \mathbf{e}_t; v) \leq \overline{\Pi}_{\Sigma t+1}^{k-1}(x_{t+1}, \mathbf{e}_t; v)$ for all $(x_{t+1}, \mathbf{e}_t)$ implies $\overline{\Pi}_{\Sigma t}^{k+1}(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v)$.) Further, by induction, $\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$ for all $k$. (Higher-than-equilibrium payoffs can be supported given higher-than-equilibrium continuation payoffs. Thus, the fact that $\overline{\Pi}_{\Sigma t}^{k-1}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$ implies $\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.)

*Part II: These upper bounds on joint welfare are non-decreasing in $x_t$.*

Base step: $k = 0$. $\overline{\Pi}_{t}^{0}(x_t, \mathbf{e}_{t-1}; v)$ is constant and hence trivially non-decreasing in $x_t$.

Induction step: $k \geq 1$. Suppose that $\overline{\Pi}_{t}^{k-1}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in $x_t$ for all $t$. Observe that, for any $x_t^H \succeq x_t^L$,

$$\begin{aligned} E[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v)|x_t^H, \mathbf{e}_t]) &= \int_0^\infty \Pr(\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z|x_t^H, \mathbf{e}_t; v)\mathrm{d}z \\ &\geq \int_0^\infty \Pr(\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z|x_t^L, \mathbf{e}_t; v)\mathrm{d}z \qquad (24) \\ &= E[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v)|x_t^L, \mathbf{e}_t]) \end{aligned}$$

By the induction hypothesis, $\{x_{t+1} \in \mathcal{X}_{t+1} : \overline{\Pi}_{t+1}^{k-1}(X_{t+1}; \mathbf{e}_t; v) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all $z$. Inequality (24) now follows from Assumption 4. Thus, for any given effort-history $\mathbf{e}_t$, $\max\left\{v_\Sigma, \delta E\left[\overline{\Pi}_{\Sigma t+1}^{k-1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}$ is non-decreasing in $x_t$, so that higher $x_t$ slackens the IC-constraint (23) while increasing the second term of (22). Finally, the first term of (22) is non-decreasing in $x_t$ by Assumption 2. All together, we conclude that the value of the maximization (22) is non-decreasing in $x_t$. This completes the desired induction.

Let $\overline{\Pi}_{\Sigma t}^{\infty}(x_t, \mathbf{e}_{t-1}; v)$ denote the pointwise limit of $\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v)$ as $k \to \infty$. Since

$\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in $x_t$ for all $k$, $\overline{\Pi}^{\infty}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ inherits this monotonicity as well (given the continuity of stage-game payoffs imposed by Assumption 1).

*Part III: Limit of upper bounds can be achieved in SPE.* It suffices now to show that $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) = \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. As shown earlier, $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. Let $e_t(x_t, \mathbf{e}_{t-1})$ denote a limit of any sequence of solutions to (22) subject to (23), as $k \to \infty$. (Such efforts are implicitly defined by (5-6) in the text.) By construction, and assumed continuity of stage-game payoffs, efforts $e_t(x_t, \mathbf{e}_{t-1})$ are incentive-compatible if players expect continuation play in later periods that generates time-$(t+1)$ payoffs of $\overline{\Pi}^{\infty}_{t+1}(x_{t+1}, \mathbf{e}_t; v)$ for each player. Again by construction, these efforts generate continuation payoffs $\overline{\Pi}^{\infty}_{t+1}(x_{t+1}, \mathbf{e}_t; v)$; thus, these strategies constitute a welfare-maximizing SPE. Thus, $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. This completes the proof. □

## Proof of Theorem 3

By Theorem 2, joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ in the joint welfare-maximizing SPE is weakly increasing in $x_t$ for all $(\mathbf{e}_{t-1}, v)$ . Given an exogenous stochastic process, further, such payoffs do not depend on the history of efforts. Since the outside option $v$ is held fixed, I will henceforth use the simpler notation $\overline{\Pi}^{eqm}_{\Sigma t}(x_t)$ here.

*Proof of (i).* Recall that $\overline{\Pi}^{eqm}_{\Sigma t}(x_t) = \max_{e_t}\left(\pi_{\Sigma t}(e_t; x_t) + E\left[\delta \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right]\right)$ subject to the IC-constraint $E\left[\delta \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] \geq c_{\Sigma t}(e_t; x_t) + 2v$. Joint continuation payoff

$$E\left[\delta \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] = \int_0^\infty \Pr\left(\delta \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z|x_t\right) \mathrm{d}z \tag{25}$$

is weakly increasing in $x_t$: $\{x_{t+1} : \delta \overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ so that, by Assumption 4, each of the probability terms in (25) is weakly increasing in $x_t$. Finally, since efforts do not control future payoffs, time-$t$ efforts in the optimal SPE will be chosen to maximize joint stage-game payoff subject to the IC-constraint. Since joint continuation payoff is weakly increasing in $x_t$, so is the set of effort-profiles $e_t$ satisfying the IC-constraint. Consequently, realized joint stage-game payoff is weakly increasing in $x_t$.

*Proof of (ii).* In the optimal SPE, the partnership ends in the first period in which joint continuation payoff is less than $2v$ (and joint continuation payoff depends only on the current state). Let $p_t^k(x_t)$ denote the probability that the partnership will end at time $t+k$, given that it is active at time $t$ in state $x_t$. I need to show that, for each $k \geq 1$, $p_t^k(x_t)$ is weakly increasing in $x_t$. The proof is by induction.

Base step. By Theorem 3(i), $p_{t'}^1(x_{t'})$ is weakly increasing in $x_{t'}$, including for $t' = t+k-1$.

Induction step. As the induction hypothesis, suppose that the following is true for some $t'+1 \in [t+1, t+k-1]$:

- $p_{t'+1}^m(x_{t'+1})$ is weakly increasing in $x_{t'+1}$ for all $m = 1, ..., t+k-t'-1$.

To complete the proof, it suffices to establish that $p_{t'}^m(x_{t'})$ is weakly increasing in $x_{t'}$ for all $m = 1, ..., t+k-t'$, since then we may conclude by induction that $p_t^k(x_t)$ is weakly increasing in $x_t$. (The argument applies for all $k \geq 1$.)

First, Theorem 3(i) implies that $p_{t'}^1(x_{t'})$ is weakly increasing in $x_{t'}$. For all $m > 0$,

$$p_{t'}^m(x_{t'}) = p_{t'}^1(x_{t'}) E\left[p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) \middle| x_{t'}\right] \tag{26}$$

(The partnership survives for $m$ periods iff it survives for $m-1$ periods after first surviving for one period.) The base step showed that the first term of (26) is weakly increasing in $x_{t'}$. It suffices now to show the same of the expectation term

$$E\left[p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) \middle| x_{t'}\right] = \int_0^1 \Pr\left(p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) > p \middle| x_{t'}\right) \mathrm{d}p \tag{27}$$

By the induction hypothesis, each set $\{x_{t'+1} \in \mathcal{X}_{t'+1} : p_{t'+1}^{m-1}(x_{t'+1}; x_{t'}) > p\}$ is both an increasing subset of $\mathcal{X}_{t'+1}$ and weakly increasing in $x_{t'}$ (relative to set inclusion). By Assumption 4, we conclude that each of the probability-terms in (27) is weakly increasing in $x_{t'}$. This completes the proof. □

## Proof of Theorem 4

As argued in the text, no joint outside option greater than $v_\Sigma^*$ can possibly be supported in partnership-economy equilibrium. To complete the proof, it suffices to verify that the strategies specified in Theorem 4 constitute a SPE and generate outside options $v_\Sigma^*$. (The theorem specifies play on the equilibrium path; augment this with shirking and quitting to start a fresh relationship should either player deviate from this path of play.)

Let $p^* = \frac{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;v^*_\Sigma)\right]-2m-v^*_\Sigma/\delta}{E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;v^*_\Sigma)\right]-v^*_\Sigma}$ be the probability with which players shirk and quit immediately based on the public randomization. Note that, by construction,

$$p^* v^*_\Sigma + (1-p^*)E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;v^*_\Sigma)\right] - 2m = \frac{v^*_\Sigma}{\delta}. \tag{28}$$

Thus, if players adopt the specified strategies, economy-wide play generates ex ante expected joint payoff $\frac{v^*_\Sigma}{\delta}$ at birth, supporting the maximal joint outside option $v^*_\Sigma$. It suffices now to show that these strategies constitute a SPE. First, players are willing to shirk and quit when the public randomization is less than $p^*$, since they expect uncooperative continuation play in the current relationship. Second, given joint outside option $v^*_\Sigma$, Theorem 1 specifies SPE continuation play should the public randomization be more favorable. This completes the proof. □

## Proof of Claim 2

*Proof. Part I: Preliminaries.* Since $x_0 = (s_o, y_0)$, continuity of $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0;v_\Sigma)]$ in $v_\Sigma$ follows from continuity of $E[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0,y_0;v_\Sigma)]$ for all $y_0$. Thus, without loss I will focus on the special case in which the initial state is simply the partnership type, $X_0 = S_0$.

Maximal SPE ex ante expected joint payoff may be expressed as

$$E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0,;v_\Sigma)\right] = \sum_{t\geq 0}\delta^t\Bigg(\Pr(T\geq t)E\left[\pi_{\Sigma t}(e_t(H_t;v_\Sigma);h_t)|T\geq t\right] + v_\Sigma \Pr(T=t)\Bigg)$$

where $T$ is the random stopping time of the partnership in the optimal SPE (which depends on $v_\Sigma$), and $e_t(h_t;v_\Sigma)$ is the prescribed effort-profile in the optimal SPE in history $h_t = (x_t, \mathbf{e}_{t-1})$. Recall that $e_t(h_t;v_\Sigma)$ maximizes joint payoff subject to the IC-constraint that joint continuation payoff is greater than or equal to joint outside option plus joint cost of effort:[37]

$$c_{\Sigma t}(e_t(h_t);h_t) \leq \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t;v_\Sigma), X_{t+1};v_\Sigma)\right] - v_\Sigma \tag{29}$$

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[37] To simplify the presentation, I focus on the case in which there is a unique such maximizer at almost all histories reached on the equilibrium path. More generally, the proof extends almost unchanged, when one recognizes that a discontinuity of $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0,;v_\Sigma)\right]$ in $v_\Sigma$ requires that the IC-constraint be binding on *all* such maximizers at a set of histories reached with positive probability.

An increase in joint outside option from $v_\Sigma$ to $v_\Sigma + \varepsilon$ has two effects on the maximal SPE joint payoff. First, the direct effect is that players enjoy higher joint payoff when quitting and quit whenever they were previously almost indifferent to doing so. This direct effect increases joint payoff by at most $\varepsilon$. Second, since $\delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t; v_\Sigma), X_{t+1}; v_\Sigma)\right] - v_\Sigma$ is strictly decreasing in $v_\Sigma$ (see the proof of Theorem 4), an indirect effect is that players cannot support as many effort-profiles at some histories. This decreases payoffs at those histories, inducing more exit and less effort at previous histories, and so on in a backward cascade that decreases joint payoff and may do so discontinuously.

*Part II: (29) is binding with zero probability.* Fix any joint outside optin $v_\Sigma$, effort-profile $e_t$, effort-profile history $\mathbf{e}_{t-1}$, and sequence of states $\mathbf{x}_{1\to t} = (x_1, ..., x_t)$ realized after time 0. By assumption, $c_{it}(e_t; s_0, \mathbf{x}_{1\to t})$ is strictly decreasing in $s_0$ for each player $i$ while, by the proof of Theorem 2, $E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1}, e_t, X_{t+1}; v_\Sigma)\right]$ is weakly increasing in $s_0$. Thus, if the IC-constraint (29) binds for some efforts $e_t$ at history $(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$, then for all $s_0^l < s_0 < s_0^h$ it fails at history $(s_0^l, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$ and is strictly satisfied at history $(s_0^h, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$. In particular, let $e_t(h_t)$ denote the effort-profile prescribed in the joint welfare-maximizing SPE from history $h_t$. Then (29) binds on $e_t(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$ for a zero-measure set of types $s_0 \in \mathbf{R}$. We conclude that, with probability one in the joint welfare-maximizing SPE, the IC-constraint will not be binding on *any* effort-profile prescribed on the equilibrium path.

*Part III: Non-binding (29) implies that maximal SPE joint payoff is continuous in $v_\Sigma$.* I will prove right-continuity here, that $\lim_{\varepsilon\to 0} \overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma + \varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ for all $v_\Sigma$ and all histories $h_t$ reached with probability one on the equilibrium path. The proof of left-continuity is similar, and omitted to save space.

For this final step, I employ a variation on the algorithm used in the proof of Theorem 2. Fix $\hat{v}_\Sigma$. For all histories $h_t$ and $\varepsilon \geq 0$, define

$$\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma) + \varepsilon$$

Since the positive "direct effect" of higher joint outside option discussed earlier is at most $\varepsilon$ and the "indirect effect" is always negative, $\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon) > \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon)$. Also, clearly, $\overline{\Pi}^{1}_{\Sigma t}(h_t; v_\Sigma)$ is continuous in $v_\Sigma$ for all $t$ and all histories $h_t$.

Next, as in Steps A-C of the algorithm illustrated in Figure 5, define

$$\begin{aligned}
\overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma) &= \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^1_{\Sigma t+1}(h_t, e_t, X_{t+1}; v_\Sigma)|h_t, e_t\right]\right\} \\
\mathcal{F}^2_t(h_t; v_\Sigma) &= \left\{e_t : v_\Sigma + c_{\Sigma t}(e_t; x_t) \le \overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)\right\} \\
\overline{\Pi}^2_{\Sigma t}(h_t; v_\Sigma) &= \max_{e_t \in \mathcal{F}^2_t(h_t; v_\Sigma)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)\right)
\end{aligned}$$

As already shown, $\overline{\Pi}^1_{\Sigma t+1}(h_t, e_t, x_{t+1}; v_\Sigma)$ is continuous in $v_\Sigma$ for all histories $h_{t+1} = (h_t, e_t, x_{t+1})$. Thus, $\overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)$ is continuous in $v_\Sigma$ for all $t, h_t$ as well. Next, since the set of effort-profiles is finite, Part II above implies that the IC-constraint is not (exactly) binding for *any* effort-profile at a probability-one set of histories reached on the equilibrium path. At all such histories, $\mathcal{F}^2_t(h_t; v_\Sigma)$ is unchanging in a neighborhood of $\hat{v}_\Sigma$. We conclude that, at a probability-one set of equilibrium histories, $\overline{\Pi}^2_{\Sigma t}(h_t; v_\Sigma)$ is continuous in $v_\Sigma$ at $\hat{v}_\Sigma$.

Repeating this argument for all $k \ge 1$, we conclude that $\overline{\Pi}^k_{\Sigma t}(h_t; v_\Sigma)$ is continuous in $v_\Sigma$ at $\hat{v}_\Sigma$ at a probability-one set of equilibrium histories. Such continuity carries over to the limit as well, so that maximal equilibrium joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ is continuous in $v_\Sigma$ at a probability-one set of histories. In particular, $E[\overline{\Pi}^{eqm}_{\Sigma t}(S_0; v_\Sigma)$ is continuous in $v_\Sigma$ at $\hat{v}_\Sigma$. ☐

## Proof of Claims 3 - 4

*Proof. Part I:* $v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is non-decreasing in* $v_\Sigma$. In a slight variation on the notation used in the text, let $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma)$ denote the maximal SPE joint payoff from history $h_t = (x_t, \mathbf{e}_{t-1})$ given joint outside option $v^h_\Sigma$. Consider now a lower joint outside option $v^l_\Sigma \in [0, v^h_\Sigma)$ and let $\Pi_{\Sigma t}(h_t; v^l_\Sigma)$ denote the joint payoff that would result should players with joint outside option $v^l_\Sigma$ mimic welfare-maximizing play as if it were $v^h_\Sigma$. Note that the stage-game payoff process and the partnership stopping time $T$ are identically distributed when players follow the same strategies. Thus, the only difference in payoffs arises from the fact that players only get $v^l_\Sigma$ when the partnership ends instead of $v^h_\Sigma$. In particular, for all histories $h_t$,

$$\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma) - \Pi_{\Sigma t}(h_t; v^l_\Sigma) = (v^h_\Sigma - v^l_\Sigma) \sum_{t' \ge t} \delta^{t'-t} \Pr(T = t'|h_t) \le v^h_\Sigma - v^l_\Sigma \tag{30}$$

Let $e_t(v_\Sigma^h)$ denote the efforts played in the optimal SPE given joint outside option $v_\Sigma^h$. Observe that these efforts remain incentive-compatible given lower joint outside option $v_\Sigma^l$:

$$\begin{aligned} E\left[\Pi_{\Sigma t+1}(H_{t+1}; v_\Sigma^h)|h_t, e_t(v_\Sigma^h)\right] &\geq E\left[\overline{\Pi}_{\Sigma t+1}^{eqm}(H_{t+1}; v_\Sigma^h)|h_t, e_t(v_\Sigma^h)\right] - \left(v_\Sigma^h - v_\Sigma^l\right) && (31) \\ &\geq \frac{v_\Sigma^h + c_{\Sigma t}(e_t(v_\Sigma^h); x_t)}{\delta} - \left(v_\Sigma^h - v_\Sigma^l\right) && (32) \\ &> c_{\Sigma t}(e_t(v_\Sigma^h); x_t) + v_\Sigma^l \end{aligned}$$

(31) follows from (30). (32) follows from the incentive-compatibility constraint (8) as applied to the optimal equilibrium given $v_\Sigma^h$, for any efforts having non-zero cost. (Mimicking zero-cost efforts is trivial in SPE regardless of outside options.) By similar logic, staying is incentive-compatible given these mimicking strategies whenever players stay in the optimal equilibrium given joint outside option $v_\Sigma^h$. (Details omitted to save space.) Thus, these mimicking strategies constitute a SPE given $v_\Sigma^l$. In particular, $E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma^l)] \geq E[\Pi_0(X_0; v_\Sigma^l)]$. Thus, $E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma^h)] - E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma^l)] \leq v_\Sigma^h - v_\Sigma^l$ as desired.

*Part II:* $v_\Sigma = \delta\left(E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)] - 2m\right)$ *has a unique solution* $v_\Sigma^*$. By definition of partnership-economy equilibrium, $\delta\left(E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)] - 2m\right) \geq 0$ since otherwise no partnerships would form. On the other hand, since joint stage-game payoff is uniformly bounded (again by Assumption 1), there exists some outside option $\overline{v}$ such that $\delta\left(E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)] - 2m\right) < \overline{v}$. Existence and uniqueness of a solution to $v_\Sigma = \delta\left(E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)] - 2m\right)$ now follows immediately from the fact that this difference is continuous (by assumption) and strictly increasing in $v_\Sigma$ (by Part I). Also, by this assumed continuity, this $v_\Sigma^*$ is the maximal joint outside option defined in Theorem 4. This complete the proof of Claim 3.

*Part III:* $v_\Sigma^*$ *is strictly decreasing in matching cost* $m$. Let $v_\Sigma^*(m)$ denote the maximal joint outside option as a function of $m$. By Part I, $v_\Sigma - E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)]$ is non-decreasing and hence $v_\Sigma - E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma)]$ is strictly increasing in $v_\Sigma$. Since $v_\Sigma^*(m) = \delta\left(E[\overline{\Pi}_{\Sigma 0}^{eqm}(X_0; v_\Sigma^*(m))] - 2m\right)$ by Claim 3, an increase in $m$ must therefore result in a decrease in $v_\Sigma^*(m)$. This completes the proof of Claim 4. □

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